SPACE PHYSICS ADVANCED STUDY OPTION: The Sun and the solar corona

Problem Sheet 1: Some problems related to the magnetic structures in the solar corona

It is clear that in the solar corona, magnetic structures play a major role. The following examples explore a few simple structures and their properties. All the examples are directly based on either the text or the problems in Introduction to Space Physics (edited by M. Kivelson and C. Russell), Chapter 3: The Sun and its Magnetohydrodynamics by E. Priest.

The first example is worked out here. The others are to be solved. Solutions will be distributed later in the term.

1.1. (Problem 3.7)

We have a magnetic field given as $B_x = B_o e^{-kz}\cos kx$, $B_{\boldsymbol{z}} = -B_o e^{-kz}\sin kx$ with $|\boldsymbol{x}| < \pi/2k$ and $\boldsymbol{z} > 0$. Here k is a positive (not necessarily integer) constant.

(i) Verify that $\nabla\cdot\boldsymbol{B} = 0$. (This is always a good point to start, as all magnetic field models **must** satisfy this equation of Maxwell.)

(ii) Find the general equation of field lines and sketch a number of them.

(iii) Verify that this model magnetic field has zero current.

SOLUTION

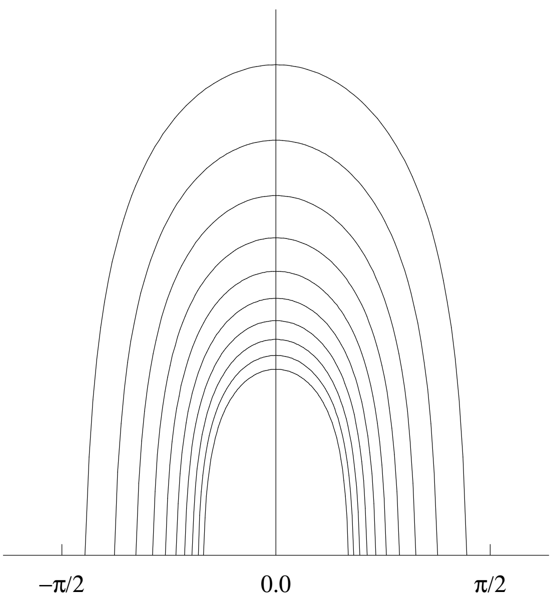
(i) We have here $B_y = 0$. So that $\nabla\cdot\boldsymbol{B} = \dfrac{\partial B_x}{\partial x} + \dfrac{\partial B_z}{\partial z} = -kB_o e^{-kz}\sin kx + kB_o e^{-kz}\sin kx = 0$

(ii) The equation of a field line is derived from $\dfrac{\boldsymbol{dx}}{\boldsymbol{B_x}} = \dfrac{\boldsymbol{dz}}{\boldsymbol{B_z}}\left(=\dfrac{\boldsymbol{dy}}{\boldsymbol{B_y}}\right)$, or, in this case, from

$\dfrac{dx}{B_o e^{-kz}\cos kx} = -\dfrac{d\,z}{B_o e^{-kz}\sin kx}$ or $\dfrac{dx}{\cos kx} = -\dfrac{dz}{\sin kx}$ which we can integrate

$$\boldsymbol{z}+\boldsymbol{integration\ constant} = -\int\frac{\sin\boldsymbol{kx}}{\cos\boldsymbol{kx}}\boldsymbol{dx} = \frac{1}{\boldsymbol{k}}\int\frac{\boldsymbol{d}(\cos\boldsymbol{kx})}{\cos\boldsymbol{kx}} = \frac{1}{\boldsymbol{k}}\ln(\cos\boldsymbol{kx})$$

where the integration constant is chosen to satisfy $\boldsymbol{z} > 0$ for $|\boldsymbol{x}| < \pi/2k$. A computer generated sketch is shown below, illustrating why this is considered to be a good, if simple model of a coronal arcade.



(iii) To show that the magnetic structure is current-free, we use Ampere's law $\nabla\times\boldsymbol{B} = \dfrac{1}{\mu_{\boldsymbol{o}}}\boldsymbol{j}$.

Because $B_y = 0$ and neither B_x nor B_z depends on y, we calculate only the y component of $\nabla\times\boldsymbol{B}$, so that

$$(\nabla\times B)_{\mathbf{y}} = \frac{\partial\mathbf{B_x}}{\partial\mathbf{z}} - \frac{\partial\mathbf{B_z}}{\partial\mathbf{x}} =$$
$$= -\mathbf{B_o k e^{-kz}}\cos\mathbf{kx} + \mathbf{B_o k e^{-kz}}\cos\mathbf{kx} = 0$$

1.2. (Based on Problems 3.2 and 3.6)

We are given a magnetic field structure as $B_x = B_o y$ and $B_y = B_o x$ (B_z is zero).

(i) Verify that $\nabla \cdot \boldsymbol{B} = 0$.

(ii) Find the equation of the magnetic field lines and sketch them.

(iii) Calculate the forces arising from the magnetic pressure and magnetic tension.

1.3. (Based on Problem 3.3)

Verify that the one dimensional magnetic diffusion equation (valid when the magnetic Reynolds number R_m <1) $\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2}$ has a solution of the form $\boldsymbol{B}(\boldsymbol{x},\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{t})\exp\left(-\boldsymbol{x}^2/4\eta\,\boldsymbol{t}\right)$. Find the differential equation which defines $\boldsymbol{f}(\boldsymbol{t})$ and determine $\boldsymbol{f}(\boldsymbol{t})$. Sketch $\boldsymbol{B}(\boldsymbol{x},\boldsymbol{t})$ for several values of $t > t_\mathrm{o}$, given that $\boldsymbol{B}(0,\boldsymbol{t_o}) = \boldsymbol{B_o}$.

1.4. (Based on Problem 3.12)

In a cylindrical coordinate system $(\boldsymbol{r},\varphi,\boldsymbol{z})$ we define an axisymmetric magnetic field, given for $t = t_\mathrm{o}$ as

$$\boldsymbol{B_z}(\boldsymbol{r},\boldsymbol{t_o}) = \boldsymbol{B_o}\exp\left(-\frac{\boldsymbol{r}^2}{4\eta\,\boldsymbol{t_o}}\right)$$

with $B_r = B_\varphi = 0$. We assume in this problem that the magnetic field was set up in a stationary plasma (velocity = 0); the plasma environment is only specified by the magnetic diffusivity $\eta = 1/\mu_{\boldsymbol{o}}\sigma$, where σ is the (finite) conductivity of the plasma.

(i) Verify that $\nabla \cdot \boldsymbol{B} = 0$.

(ii) What are the units of η ? (You can use this result later in this problem to check your calculations.)

(iii) Show that the differential equation for $\partial \boldsymbol{B}/\partial t$ applicable in these circumstances is $\frac{\partial \boldsymbol{B_z}}{\partial \boldsymbol{t}} = \frac{\eta}{\boldsymbol{r}}\frac{\partial}{\partial \boldsymbol{r}}\left(\boldsymbol{r}\frac{\partial \boldsymbol{B_z}}{\partial \boldsymbol{r}}\right)$

(iv) If we seek a solution to this differential equation in the form $\boldsymbol{B_z}(\boldsymbol{r},\boldsymbol{t}) = \boldsymbol{f}(\boldsymbol{t})\exp\left(-\frac{\boldsymbol{r}^2}{4\eta\,\boldsymbol{t}}\right)$, show that $\boldsymbol{f}(\boldsymbol{t})$ satisfies the differential equation $\frac{\boldsymbol{df}(\boldsymbol{t})}{\boldsymbol{dt}} + \frac{\boldsymbol{f}(\boldsymbol{t})}{\boldsymbol{t}} = 0$.

(v) What is therefore the solution $\boldsymbol{B_z}(\boldsymbol{r},\boldsymbol{t})$ to the differential equation in (iii) ?

(vi) Show that the total flux of the magnetic field is conserved for all $t \geq t_\mathrm{o}$.

(vii) Calculate the total magnetic energy in a slab of space 1 m high in the z (axial) direction and conclude that it is a decreasing function of time. Where is the missing energy ?

(viii) Sketch $\boldsymbol{B_z}(\boldsymbol{r},\boldsymbol{t})$ vs r for three (increasing) values of $t \geq t_\mathrm{o}$.

(ix) Sketch the ratio of magnetic energy at time t to the energy at time $t = t_\mathrm{o}$ in the slab vs t.