Space Physics Advanced Option

Answers to problems related to the magnetic structures in the solar corona

The solution to Question 1 was given in the Worksheet.

Question 2

(a) Given a magnetic field structure as $B_x = B_o y$, $B_y = B_o x$ and $B_z = 0$, we have

$\frac{\partial B_x}{\partial x} = 0$, $\frac{\partial B_y}{\partial y} = 0$ and $\frac{\partial B_z}{\partial z} = 0$, therefore $\nabla \cdot \mathbf{B} = 0$

Note already at this stage, that if B_0 is in magnetic induction units (Tessla), then the coordinates x and y must be dimensionless.

(b) The differential equation defining the magnetic field lines is $\frac{dx}{B_x} = \frac{dy}{B_y}$, or, substituting, $\frac{dx}{B_0 y} = \frac{dy}{B_0 x}$, or

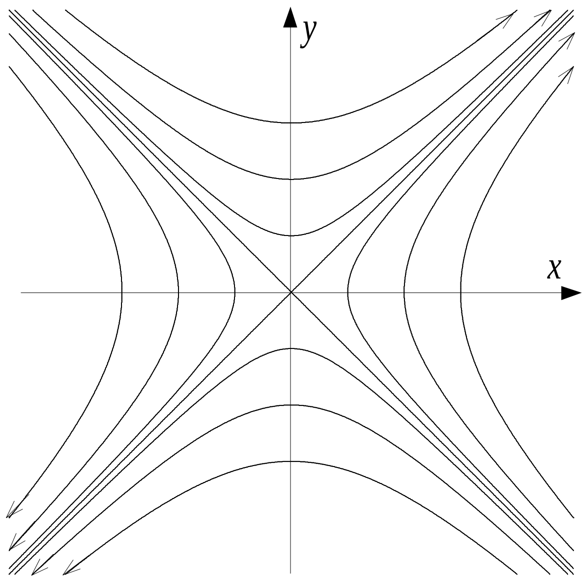
$y dy = x dx$. Integrating, we get $y^2 = x^2 + C$ where C is the integration constant. This equation defines the magnetic field lines which are families of hyperbolae. Note that for $C = 0$, we have $y = \pm x$. To see that in general the equation defines hyperbolae, perform the change of variables

$u = \frac{x}{\sqrt{2}} - \frac{y}{\sqrt{2}}$ and $v = \frac{x}{\sqrt{2}} + \frac{y}{\sqrt{2}}$.

Note that this corresponds to a clockwise rotation of the coordinate axes by 45°. Inverting these two equations gives

$x = \frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}$ and $y = -\frac{u}{\sqrt{2}} + \frac{v}{\sqrt{2}}$.

Substituting into the equation of the magnetic field lines $y^2 = x^2 + C$ gives the magnetic field lines in the new coordinate system as $v = -\frac{C}{2}\frac{1}{u}$ which are indeed hyperbolae, dependent on the value of the constant C. The magnetic field lines can be sketched as illustrated below:



(c) The magnetic pressure is defined as $p_B = \frac{B^2}{2\mu_0} = \frac{(B_0 y)^2 + (B_0 x)^2}{2\mu_0} = \frac{B_0^2}{2\mu_0}\left(x^2 + y^2\right)$

The magnetic pressure force is $-\nabla p_B = -\dfrac{B_0^2}{\mu_0}(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}})$ where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are the unit vectors along the coordinate axes.

The magnetic tension force is given by $\dfrac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} = \dfrac{1}{\mu_0}\left[\left(B_x\dfrac{\partial B_x}{\partial x} + B_y\dfrac{\partial B_x}{\partial y}\right)\hat{\mathbf{x}} + \left(B_x\dfrac{\partial B_y}{\partial x} + B_y\dfrac{\partial B_y}{\partial y}\right)\hat{\mathbf{y}}\right]$

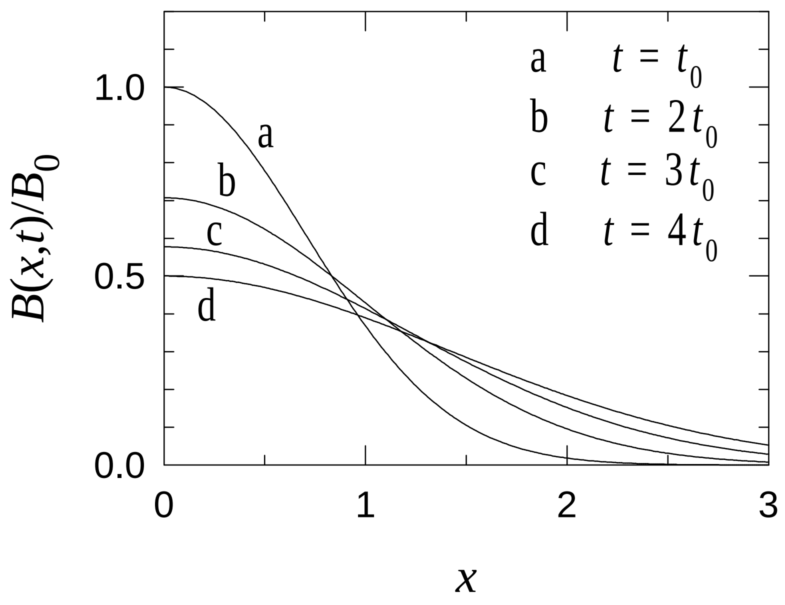
Substituting, we get $\dfrac{1}{\mu_0}(\mathbf{B}\cdot\nabla)\mathbf{B} = \dfrac{B_0^2}{\mu_0}(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}})$. Note that the magnetic pressure force is balanced by the magnetic tension in this case.

Question 3.

Given $B(x,t) = f(t)\,\mathbf{exp}(-x^2/4\eta t)$, we have

$\dfrac{\partial B}{\partial t} = \left[\dfrac{df}{dt} + f(t)\dfrac{x^2}{4\eta t^2}\right]\mathbf{exp}(-x^2/4\eta t)$ and $\eta\dfrac{\partial^2 B}{\partial x^2} = \left[-\dfrac{1}{2t} + \dfrac{x^2}{4\eta t^2}\right]f(t)\,\mathbf{exp}(-x^2/4\eta t)$ so that the equation $\dfrac{\partial B}{\partial t} = \eta\dfrac{\partial^2 B}{\partial x^2}$ is satisfied if the function $f(t)$ is a solution of the differential equation $\dfrac{df}{dt} + \dfrac{f}{2t} = 0$. Rewriting this equation as $\dfrac{df}{f} = -\dfrac{1}{2}\dfrac{dt}{t}$ we can integrate to get $\mathbf{ln}\, f(t) = -\dfrac{1}{2}\mathbf{ln}\, t + \text{constant}$ or $f(t) = B_0\sqrt{\dfrac{t_0}{t}}$.

Therefore $B(x,t) = B_0\sqrt{\dfrac{t_0}{t}}\,\mathbf{exp}(-x^2/4\eta t)$ and $B(x = 0, t = t_0) = B_0$.



Question 4.

(i) For an arbitrary vector $\mathbf{A}$ we have, in cylindrical coordinates, $\nabla\cdot\mathbf{A} = \dfrac{1}{r}\dfrac{\partial(rA_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial A_\varphi}{\partial\varphi} + \dfrac{\partial A_z}{\partial z}$. In the present case, we have $B_r = B_\varphi = 0$ and $B_z = B_z(r)$, therefore $\dfrac{\partial B_z}{\partial z} = 0$ and $\nabla\cdot\mathbf{B} = 0$.

(ii) The simplest way to identify the units of $\eta = 1/\mu_0\sigma$ is to note that in the exponential term of $B_z(r,t_0) = B_0 \,\mathbf{exp}\left(-r^2/4\eta t_0\right)$ the exponent $r^2/4\eta t_0$ is dimensionless (necessarily), and therefore as r^2/t_0 has units m^2.s^{-1} , these are also the units of η.

(iii) The magnetic induction equation is in the form of diffusion equation in this case, so that $\frac{\partial \mathbf{B}}{\partial t} = \eta \nabla^2 \mathbf{B}$, so that, as only the z component of **B** is different from zero, we can write $\frac{\partial B_z(r,t)}{\partial r} = \eta \nabla^2 B_z(r,t)$ where $B_z(r,t)$ is a scalar. The general form for an arbitrary function $g(r,\varphi,z)$ in cylindrical coordinates is

$\nabla^2 g = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial g}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 g}{\partial \varphi^2} + \frac{\partial^2 g}{\partial z^2}$. In our case, only the first term is non-zero, therefore $\frac{\partial B_z}{\partial t} = \frac{\eta}{r}\frac{\partial}{\partial r}\left(r\frac{\partial B_z}{\partial r}\right)$.

(iv) Given the form $B_z(r,t) = f(t)\,\mathbf{exp}\left(-r^2/4\eta t\right)$, we have

$$\frac{\partial B_z}{\partial t} = \left[\frac{df}{dt} + f\,\frac{r^2}{4\eta t^2}\right]\mathbf{exp}\left(-r^2/4\eta t\right)$$

$\frac{\partial B_z}{\partial r} = -f\,\frac{2r}{4\eta t}\,\mathbf{exp}\left(-r^2/4\eta t\right)$ and $\frac{\partial}{\partial r}\left(r\frac{\partial B_z}{\partial r}\right) = \left[-\frac{4r}{4\eta t} + \frac{4r^3}{(4\eta t)^2}\right] f(t)\,\mathbf{exp}\left(-r^2/4\eta t\right)$.

Substituting into the equation found in (iii) and cancelling the exponential terms on the two sides of the equation gives

$\frac{df}{dt} + f\,\frac{r^2}{4\eta t^2} = -\frac{f}{t} + f\,\frac{r^2}{4\eta t^2}$, or, as indicated in the question, $\frac{df}{dt} + \frac{f}{t} = 0$.

(v) The solution to the equation derived in (iv) can be found by integration of $\frac{df}{f} = -\frac{dt}{t}$, that is

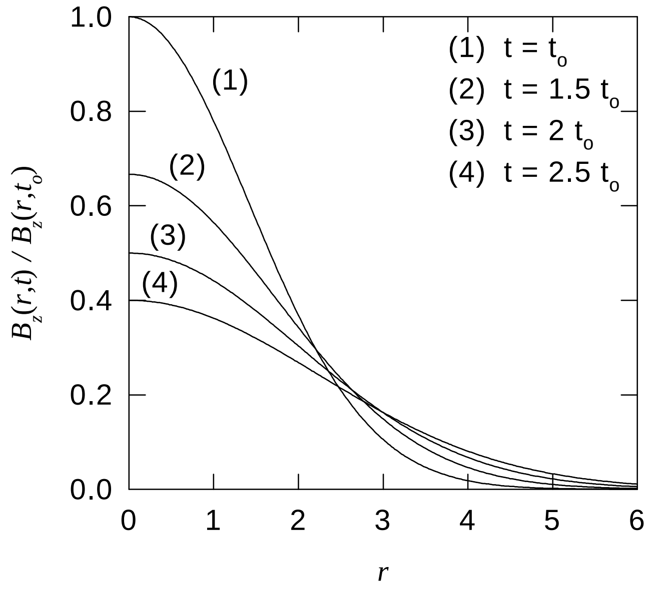
$\mathbf{ln}\, f(t) = -\mathbf{ln}\, t + \text{constant}$, or $f(t) = B_0\frac{t_0}{t}$ and therefore $B_z(r,t) = B_0\frac{t_0}{t}\,\mathbf{exp}\left(-r^2/4\eta t\right)$. Note that this solution satisfies the initial condition given in the question for $t = t_0$.

(vi) The total flux of the magnetic field is $\int_0^{2\pi} d\varphi \int_0^{\infty} rB_z\,dr = 2\pi\int_0^{\infty} rB_z\,dr = \mathbf{lim}(r \to \infty)\left[2\pi\int_0^{r} rB_z\,dr\right]$. As the general formula is $\int_0^{x} t\,\mathbf{exp}\left(-\beta t^2\right)dt = \frac{1}{2\beta}\left[1 - \mathbf{exp}\left(-\beta x^2\right)\right]$, its application here gives for the total magnetic flux

$4\pi B_0 t_0 \eta\,\mathbf{lim}(r \to \infty)\left[1 - \mathbf{exp}\left(-r^2/4\eta t\right)\right] = 4\pi B_0 t_0 \eta$, independent of time. Note that the flux is in units of Tessla.s.(m^2.s^{-1}) = Webers, correctly.

(vii) The magnetic energy in the slab is $\int_0^{2\pi} d\varphi \int_0^{\infty} r\frac{B_z^2}{2\mu_0}dr$. Using the integration technique as in (vi), we get for the total energy in the slab $\frac{\pi B_0 t_0^2 \eta}{\mu_0 t}$. This is indeed decreasing, inversely proportional to *t*. The magnetic energy that has diffused is turning into Joule heat, through the electrical conductivity of the medium.

(viii) The plot is shown on the next page.



(ix) The total energy in the slab decreases inversely with *t*, so the sketch is of the curve t_0/t vs. *t*, as shown below.