mjt Dec. 2002

Space Physics Advanced Option: Waves & Seismology of the Sun

Outline solutions to the Problem Sheet

1. Fluid equ^{n}s:

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}.\nabla\underline{u}\right) = -\nabla p + \rho \underline{g} \qquad (1) \qquad \text{momentum eq}^{n}$$

$$\frac{\partial \rho}{\partial t} + \nabla.(\rho\underline{u}) = 0 \qquad (2) \qquad \text{mass eq}^{n}$$

$$\left(\frac{\partial p}{\partial t} + \underline{u}.\nabla p\right) = c^2\left(\frac{\partial \rho}{\partial t} + \underline{u}.\nabla\rho\right) \qquad (3) \qquad \text{adiabatic energy eq}^{n}$$

Linearized forms:

In equilibrium $\nabla p_0 = \rho_0 \underline{g}$

$$\rho_0 \frac{\partial \underline{u}}{\partial t} = -\nabla p' + \rho' \underline{g} \qquad (1')$$

$$\frac{\partial \rho'}{\partial t} + \nabla.(\rho_0 \underline{u}) = 0 \qquad (2')$$

$$\frac{\partial p'}{\partial t} + \underline{u}.\nabla p_0 = -\rho_0 c_0^2 \nabla.\underline{u} \qquad (3') \qquad \text{using (2)}$$

Write $\underline{u} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ & seek perturbations $\propto e^{i\omega t}$ so $\frac{\partial}{\partial t} \to i\omega$:

$$
\begin{aligned}
i\omega\rho_0 u &= -\partial p'/\partial x + &&(1''a)\\
i\omega\rho_0 v &= -\partial p'/\partial y &&(1''b)\\
i\omega\rho_0 w &= -\partial p'/\partial z + \rho' g &&(1''c)\\
i\omega\rho' &= -\rho_0 \chi - w\, d\rho_0/dz &&(2'')\\
i\omega p' + \rho g w &= -\rho_0 c_0^2 \chi &&(3'')
\end{aligned}
$$

where $\chi \equiv \nabla.\underline{u}$. Note that sheet drops zero subscript on equil^{m} quantities.

Required results follow. E.g. $\frac{i\omega}{\rho_0}(1''c)$ & $(2'')$ & $(3'') \Rightarrow \omega^2 w = \ldots$.

If perturbations $\propto e^{ikx}$ & are independent of y, then $(1''b) \Rightarrow v = 0$. Also

$$\chi \equiv \nabla.\underline{u} = iku + \partial w/\partial z$$

$$
\begin{aligned}
\nabla\wedge\underline{u} &\equiv \left(\partial w/\partial y - \partial v/\partial z,\ \partial u/\partial z - \partial w/\partial x,\ \partial v/\partial x - \partial u/\partial y\right)\\
&= \left(0,\ \partial u/\partial z - ikw,\ 0\right) \equiv (0, \eta, 0)
\end{aligned}
$$

Also (1''a) can be rewritten

$$i\omega \rho_0 u = -ikp' \qquad (1'''a)$$

Results follow with some algebra.

(b) Also algebra, using the above. Note that

$$N^2 = g\left(-\frac{1}{\gamma} d\ln p/dz + d\ln\rho/dz\right) = g\left(-g/c^2 + \frac{d\ln\rho}{dz}\right)$$

Given equation is of form $\chi'' + a\chi' + b\chi = 0$ $\left(' \equiv \frac{\partial}{\partial z}\right)$

Want an equation with no 1st-derivative term.

Write $\chi = v.\Psi$: $\chi' = v\Psi' + v'\Psi$; $\chi'' = v\Psi'' + 2v'\Psi' + v''\Psi$.

∴ substituting

$$v\Psi'' + (2v' + av)\Psi' + (v'' + av' + bv)\Psi = 0$$

Take v to be any function that makes $2v' + av \equiv 0$

Our $a \equiv \frac{d}{dz}(\ln \rho c^4)$, so a suitable v by integration is $v = \rho^{-1/2} c^{-2}$.

Then $\Psi'' + \left(\frac{v''}{v} + a\frac{v'}{v} + b\right)\Psi = 0$.

Simplifying, get

$$\Psi'' + K^2\Psi = 0 \qquad (*)$$

where $K^2 = \dfrac{\omega^2 - \omega_c^2}{c^2} - k^2\left(1 - \dfrac{N^2}{\omega^2}\right)$

with $\omega_c^2 = \dfrac{c^2}{4H^2}\left(1 + 2\dfrac{dH}{dz}\right)$

& $H = \left(\dfrac{d\ln\rho}{dz}\right)^{-1} \equiv \rho\Big/\dfrac{d\rho}{dz}$ is the density scale height.

ω_c is the "acoustic cut-off frequency".

2. $\Psi'' + K^2\Psi = 0$

WKB theory: for K^2 large & slowly varying, get approximately sinusoidal solutions. Suppose $K^2 > 0$ for $z_1 < z < z_2$ & $K^2 < 0$ outside. Then in $z_1 < z < z_2$ $\Psi \simeq A\cos\left(\int^z K\,dz\right)$ whereas Ψ is exponentially decaying outside. $K^2(z_1) = K^2(z_2) = 0$: z_1 & z_2 are the turning points. Matching at z_1, z_2 of the two sol^{ns} ⇒ phase at $z = z_1, z_2$. Turns out phase at z_1, z_2 must be $\pi/4$, hence

$(n-\tfrac{1}{2})$ wavelengths must fit in between;

i.e., $\displaystyle\int_{z_1}^{z_2} K\,dz = (n-\tfrac{1}{2})\pi \quad (+)$, n integer.

Polytrope: $p \propto \rho^{1+1/\mu}$ (μ: polytropic index) $\qquad p = K\rho^{1+1/\mu}$ say.

Hydrostatic equilibrium $\quad dp/dz = \rho g$

$$\therefore \quad \left(1+\frac{1}{\mu}\right) K\rho^{1/\mu}\frac{d\rho}{dz} = \rho g$$

Γ_1 & γ are the same thing

$$\therefore \quad K\rho^{1/\mu} = \frac{gz}{\mu+1} + \text{constant}$$

zero to make $\rho = 0$ at $z = 0$.

$\therefore\ \rho \propto z^{\mu}$ & $p \propto z^{\mu+1}$. Also $\displaystyle c^2 = \Gamma_1 p/\rho = \Gamma_1 K\rho^{1/\mu} = \frac{\Gamma_1 g z}{\mu+1}$.

Density scale height $\displaystyle H = \left(\frac{d\ln\rho}{dz}\right)^{-1} = z/\mu$.

(Buoyancy frequency)^2 $\displaystyle N^2 = g\left(-g/c^2 + \frac{1}{H}\right) = \frac{g}{z}\left(\mu - \frac{\mu+1}{\Gamma_1}\right)$

(Cut-off frequency)^2 $\displaystyle \omega_c^2 = \frac{c^2}{4H^2}(1+2H') = c^2\frac{\mu^2}{4z^2}\left(1+\frac{2}{\mu}\right)$

$\therefore$ From (*) & (+):

$$\int_{z_1}^{z_2}\left(\frac{A}{z} - \frac{B}{z^2} - k^2\right)^{1/2} dz = (n-\tfrac{1}{2})\pi$$

where $\displaystyle A = (\mu+1)\omega^2/\Gamma_1 g + k^2 g\left(\mu - \frac{\mu+1}{\Gamma_1}\right)/\omega^2$

$\displaystyle B = \frac{\mu}{4}(\mu+2)$.

Using the hint, get

$$(n-\tfrac{1}{2}) = \frac{(\mu+1)\omega^2}{2k\Gamma_1 g} + \frac{kg\left(\mu - \frac{\mu+1}{\Gamma_1}\right)}{2\omega^2} - \sqrt{\frac{\mu}{4}(\mu+2)}$$

& so the result follows, with $2\alpha = \sqrt{\mu(\mu+2)} - 1$.

p-mode frequency is the larger of the two roots of the quadratic for σ^2:

$$\sigma^2 = \frac{2(n+\alpha) + \sqrt{4(n+\alpha)^2 - 4\frac{(\mu+1)}{\Gamma_1}\left(\mu - \frac{\mu+1}{\Gamma_1}\right)}}{2(\mu+1)/\Gamma_1}$$

For large n: $\displaystyle \sigma^2 \equiv \frac{\omega^2}{gk} \simeq \frac{4(n+\alpha)}{2(\mu+1)/\Gamma_1}$

$$\therefore \quad \omega^2 \simeq \frac{2(n+\alpha)\Gamma_1 g k}{(\mu+1)}$$

Note $\omega^2 \propto k$.

3. For radial oscillations, cannot treat $\underline{g}$ as constant. Write $\underline{g} = -\nabla\psi$ where $\nabla^2\psi = 4\pi G\rho$ — Poisson's equation.

$$\frac{\partial \underline{u}}{\partial t} = -\frac{1}{\rho}\nabla p - \nabla\psi$$

Linearize:
$$\frac{\partial \underline{u}}{\partial t} = -\frac{1}{\rho_0}\nabla p' + \frac{\rho'}{\rho_0^2}\nabla p_0 - \nabla\psi'$$

and so
$$\frac{\partial^2 \underline{u}}{\partial t^2} = -\frac{1}{\rho_0}\nabla\frac{\partial p'}{\partial t} + \frac{1}{\rho_0^2}\frac{\partial \rho'}{\partial t}\nabla p_0 - \nabla\frac{\partial\psi'}{\partial t} = 0 . \quad (\dagger\dagger)$$

Eliminate $\partial p'/\partial t$, $\partial\rho'/\partial t$ using equ^{ns} (2'), (3') of Q.1.

Trick now is that one can integrate $\frac{\partial}{\partial t}$(Poisson's equ^n) for <u>radial</u> oscillations.

$$\nabla^2\frac{\partial\psi'}{\partial t} = -4\pi G\,\nabla\cdot(\rho_0\underline{u})$$

$$\Rightarrow \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\left(\frac{\partial\psi'}{\partial t}\right)\right) = -\frac{4\pi G}{r^2}\frac{\partial}{\partial r}\left(r^2\rho u\right)$$

$$\Rightarrow \quad r^2\frac{\partial}{\partial r}\left(\frac{\partial\psi'}{\partial t}\right) = -4\pi G r^2\rho u + f(t)$$

"constant" of integration.

True $\forall r$ including $r=0$ $\Rightarrow$ $f(t)\equiv 0$.

Now substitute $\frac{\partial}{\partial r}\left(\frac{\partial\psi'}{\partial t}\right) = -4\pi G\rho u$ in place of $\nabla\frac{\partial\psi'}{\partial t}$ in $(\dagger\dagger)$

& the rest is just algebra, remembering that

$$g = Gm/r^2 \quad \& \quad dm/dr = 4\pi r^2\rho .$$

Bdy cond^n at $r=R$.

For simplicity, treat Γ_1 as constant. Then given equ^n can be manipulated into form
$$\frac{\Gamma_1 r^3}{Gm}\frac{p}{\rho}\left(r\xi'' + 4\xi'\right) = r\left[\Gamma_1 r\xi' - \left\{\frac{\omega^2 r^3}{Gm} - (3\Gamma_1 - 4)\right\}\xi\right]$$

At $r=R$ $(m=M)$, $p/\rho\to 0$ so LHS vanishes & so RHS must too

$\therefore$ a bdy cond^n is $\frac{d\xi}{dr} = \left\{\frac{\omega^2R^3}{GM} - (3\Gamma_1 - 4)\right\}\xi$ at $r=R$.

Bdy cond^n at $r=0$.

Seek Frobenius sol^n for ξ: $\xi = a_0 r^\alpha + a_1 r^{\alpha+1} + a_2 r^{\alpha+2} + \dots$.

Note also $p \sim p_0 - p_2 r^2$ (no r^1 term by hydrostatic equ^n & $m\propto r^3$)

& $\rho\sim\rho_c$ near $r=0$.

Given equ^n yields $\Gamma_1 p_0\alpha(\alpha+3)a_0 r^{\alpha+2} + \Gamma_1 p_0(\alpha+1)(\alpha+4)a_1 r^{\alpha+3} + O(r^{\alpha+4}) = 0$

Each coeff. of $r^{()}$ must vanish. First $\Rightarrow$ $\alpha=0$ or $\alpha=-3$ — reject $\alpha=-3$ which would make ξ singular at $r=0$.

$\therefore \alpha=0$. Then $r^{\alpha+3}$ term vanishing $\Rightarrow$ $a_1=0$

$\therefore$ $\xi = a_0 + a_2r^2 + \dots$. $\xi' = 0$ at $r=0$.

A bdy cond^n at $r=0$ is therefore $\frac{d\xi}{dr} = 0$ at $r=0$.

Have the 2nd order ODE for ξ + bdy cond^{n}s at $r=0, R$.
Equation & boundary conditions don't specify normalization of ξ
$\therefore$ without loss of generality $\xi = 1$ at $r = R$ say. Fix ω.
Integrate ξ inwards (say) from $r=R$, satisfying $r=R$ boundary condition. When reach $r=0$, the central boundary cond^{n} will only be satisfied for certain values of ω — these are the eigenvalues of the problem.
As ω^2 increases, ξ becomes more oscillatory in space.

4. First part: book work.
Second part: Sorry, there is one vital piece of information still required to be given, namely that the observed period of the oscillation is ~ 12 min $= 720$ seconds.
Taking $\lambda = 2 \times$ loop length $= 10^8$ m.

Fast / Alfvén	Slow
$\omega \simeq v_A k$	$\omega = c_{slow} k$
or $v_A = f\lambda$	$c_{slow} = f\lambda$

$$f\lambda = \lambda / T = \frac{10^8}{7.2 \times 10^2} \simeq 1.4 \times 10^5 \ \mathrm{m\,s^{-1}}$$

Given $c_{slow} \sim 1.5 \times 10^5 \ \mathrm{m\,s^{-1}}$, $v_A \sim 10^6 \ \mathrm{m\,s^{-1}}$,
the oscillation is probably a slow mode.