Space Physics Handout 3 : Motion of charged particles in magnetic and electric fields

We discussed the motion of a charged particle's in prescribed electric and magnetic fields during lectures. In this handout are reproduced some of the most important figures which shown during these lectures.

We showed that on the assumption of a constant and homogenous magnetic field along the z direction, the equations of particle velocity and position reveal that the particle motion in the (x, y) plane is circular, but that in three-dimensions the complete trajectory describes a helix wound around $\underline{B}$, as seen in figure 1.

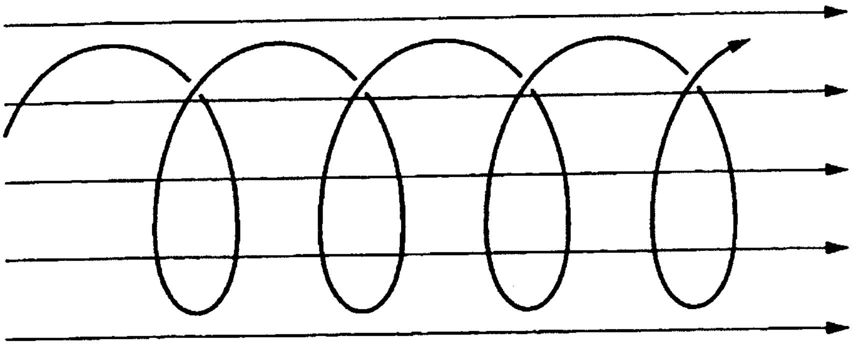


Figure 1: Helicoidal ion orbit in a constant magnetic field (Baumjohann and Treumann, 1999).

The radius of the circle in the (x-y) plane is known as the gyro-radius and defined as:

$$r_L = \frac{v_\perp}{|\Omega|} = \left|\frac{m v_\perp}{qB}\right|$$

The table below provides some typical values of the gyrofrequency and the gyroradius for three space plasma environments for protons (p) and electrons (e). (taken from Cravens, 1997)

Location	v_p	v_e	B	Ω_p	Ω_e	$R_{L,p}$	$R_{L,e}$
Solar wind	50 km/s	1000km/s	5 nT	0.5 s^{-1}	10^3 s^{-1}	100 km	1 km
Inner magneto-sphere at 3 R_E	4000km/s	5×10^4 km/s	10^3 nT	100 s^{-1}	2×10^5 s^{-1}	40 km	300 m
Ionosphere	5 km/s	200 km/s	3×10^4 nT	3000 s^{-1}	5×10^6 s^{-1}	2 m	5 cm

Guiding centre approximation

We also discussed the concept of guiding centre motion, with the motion of the particle being the vector sum of its gyration around the magnetic field, and a motion of its gyro- or guiding centre.

$$\underline{r}(t) = \underline{r}_c(t) + \underline{r}_L(t)$$

The guiding centre motion is due to the (constant) velocity of the particle, arising from the initial conditions, when the magnetic field is constant and homogenous. We found that the motion of the particle around its (instantaneous) guiding centre (given by $\underline{r}_c(t)$) is always perpendicular to the magnetic field. The motion of the gyro-centre perpendicular to the magnetic force is also perpendicular to the applied force -this is the drift velocity of the particle. We also noted that the direction of the drift motion depends on the charge of the particle (if the applied force is independent of charge), and in a plasma which consists of positively charged ions and electrons, the ions and electrons drift in opposite directions, giving rise to a current perpendicular to both the magnetic field and the applied force.

We examined an application of this drift motion in the lectures where we determined the motion of a charged particle in the presence of homogenous magnetic and electric fields. In this case, the particle's velocity perpendicular to the magnetic field is:

$$\underline{v}_\perp = \left(\frac{d\underline{r}_c}{dt}\right)_\perp + \left(\frac{d\underline{r}_L}{dt}\right)_\perp = \frac{\underline{E}\times\underline{B}}{B^2} + \frac{d\underline{r}_L}{dt}$$

This consists of two components, the first term on the right hand side is a drift term (called the $\underline{E} \times \underline{B}$ drift) which is perpendicular to both $\underline{E}$ and $\underline{B}$; the second term represents the gyration of the particle around its instantaneous guiding centre. Note the $\underline{E} \times \underline{B}$ drift is independent of the sign of the particle's charge and so electrons and ions drift in the same direction (and at the same velocity) and therefore the $\underline{E} \times \underline{B}$ drift does not generate a current. Figure 2 shows schematically how ions and electrons move in a uniform magnetic field if an electric field perpendicular to the magnetic field is also present. The direction of the gyratory motion depends on the sign of the particle's charge and the radius of the gyratory circle will vary with particle mass and would therefore be much larger for an ion than for and electron if their velocities were the same. What occurs is that the electrical force accelerates the particle during part of its orbit and decelerates it during the remaining part of the orbit. As a result the orbit is a distorted circle, with a larger than average radius of curvature during half of the orbit and a smaller radius of curvature during the remaining half of the orbit.

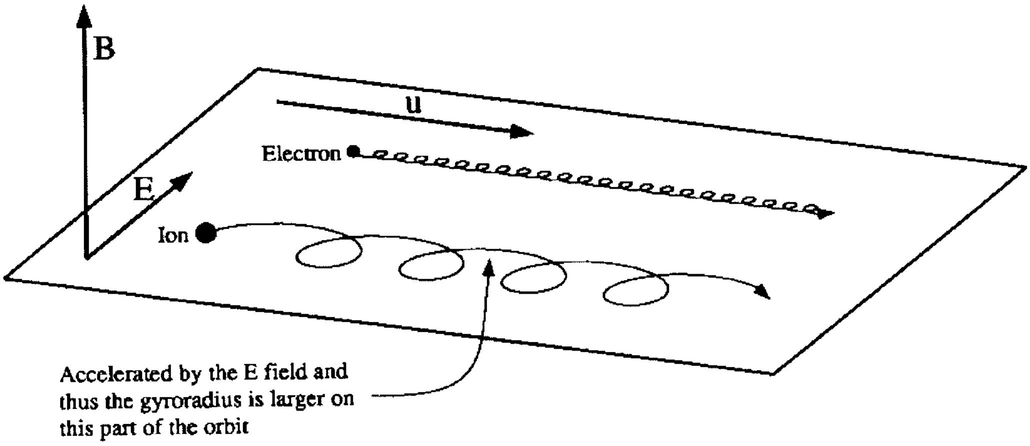


Figure 2: Schematic of ion and electron motion (taken from Kivelson and Russell, 1997)

A similar effect is observed for particle motion in a changing magnetic field, where here ∇B causes the particle's instantaneous gyroradius to change size. In the case of a non-uniform magnetic field, the particle's then drift under the effect of the magnetic field gradient, with a drift velocity of

$$\underline{v}_{grad} = \frac{m v_{\perp}^2}{2qB^3} \left(\underline{B} \times \nabla \underline{B}\right)$$

This is the gradient drift motion of the particle. If, in the non-uniform magnetic field the magnetic field lines are curved, then the particle undergoes a curvature drift, with a velocity of:

$$\underline{v}_{CD} = \frac{m v_{\parallel}^2}{q} \frac{\underline{R}_c \times \underline{B}}{R_c^2 B^2}$$

Where Rc is the local radius of curvature and $v_{\parallel}$ is the component of the particle velocity parallel to the magnetic field, see figure 3. Both the gradient and the curvature drifts depend on the sign of the charge of the particle, so that electrons and positively charged ions drift in opposite directions, giving rise to a current.

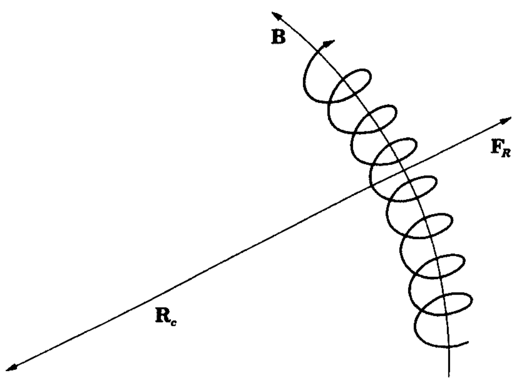


Figure 3 : Centrifugal force felt by a particle moving along a curved magnetic field line (taken from Baumjohann and Treumann, 1999).

Appendix

The information which follows derives from basics the curvature drift of a particle, which we referred to in lectures, as well as providing additional information on general curved motion for those of you interested.

Derivation from basics of the curvature drift of a particle

Circular motion

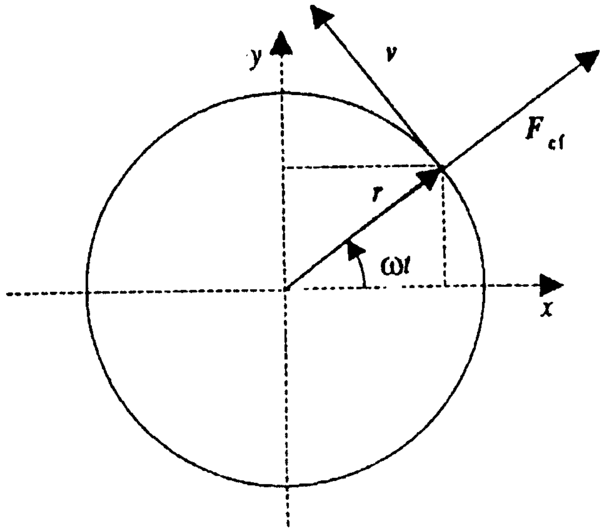
We assume a particle of mass m, moving at a constant velocity v_0 on a circle of radius r_0 and with a constant angular velocity $\omega = v_0/r_0$. The instantaneous position of the particle is defined by its position vector:

$$\underline{r}(t) = \hat{x}x + \hat{y}y = \hat{x}r_0\cos\omega t + \hat{y}r_0\sin\omega t$$

The velocity vector of the particle is: $\underline{v}(t) = \hat{x}\frac{dx}{dt} + \hat{y}\frac{dy}{dt} = -\hat{x}\omega r_0\sin\omega t + \hat{y}\omega r_0\cos\omega t$

And the acceleration of the particle is: $\underline{a}(t) = \hat{x}\frac{d^2x}{dt^2} + \hat{y}\frac{d^2y}{dt^2} = -\hat{x}\omega^2 r_0\cos\omega t - \hat{y}\omega^2 r_0\sin\omega t$

The particle is moving under the influence of the centripetal force (pointing from the particle to the centre of motion). This is balanced by the centrifugal force $\underline{F}_{cf}$, pointing away from the centre of motion, along the instantaneous radius vector of the particle: $\underline{F}_{cf} = -m\frac{d\underline{v}}{dt} = -m\frac{d^2\underline{r}}{dt^2} = m\omega^2\underline{r} = m\frac{v_0^2}{r_0^2}\underline{r}$



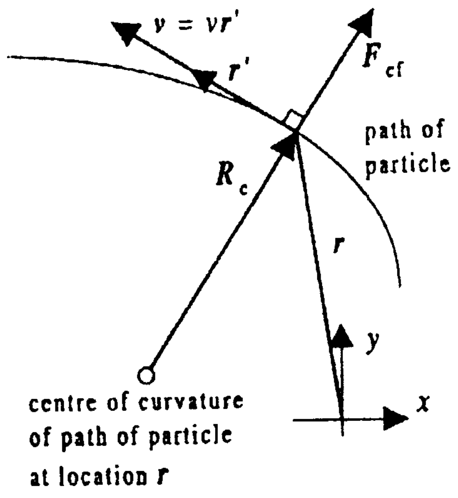
General curved motion

For a general curved motion we need to calculate the instantaneous radius of curvature. For this we need to use the characteristics of general 3D curves. If the position vector of the particle is given by:

$$\underline{r}(t) = x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}$$

Then we can define a distance parameter along the curve (the path of the particle) as :

$$s(t) = \int_{t_0}^{t}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right]^{1/2} dt$$



Taking the positive square root means that the particle moves in the direction of increasing s. In the following we denote differentiation with respect to s (≡ d/ds) with a dash:

$$\underline{r}' = \frac{d\underline{r}}{ds} = \frac{d\underline{r}}{dt}\frac{dt}{ds}$$

As, $d\underline{r}/dt = \underline{v}$ is the velocity vector of the particle, and $ds/dt = |v| = v$ is the magnitude of the velocity vector , we have:

$$\underline{r}' = \frac{\underline{v}}{v}$$

which is the unit vector along the tangent to the path of the motion (and pointing in the direction of the motion of the particle that is moving in the direction of increasing s). As $\underline{r}'$ is a unit vector, we have $\underline{r}'.\underline{r}' = 1$
On differentiating this with respect to s, we get $2\underline{r}'.\underline{r}'' = 0$

This defines the vector $\underline{r}'' = \dfrac{d^2\underline{r}}{ds^2}$ as being perpendicular to $\underline{r}'$,

therefore perpendicular to the velocity vector $\underline{v}$ of the particle. The vector $\underline{r}''$ defines the principal normal to the 3D curve of the path of the particle.

We can define a unit vector along the principal normal as $\underline{\hat{n}} = \dfrac{\underline{r}''}{|\underline{r}''|} = R_c\,\underline{r}''$

where $\dfrac{1}{R_c} = |\underline{r}''|$

is the instantaneous curvature of path of the particle and R_c is the radius of curvature.

We also have $\underline{r}'' = \dfrac{d\underline{r}'}{ds} = \dfrac{d}{dt}\dfrac{dt}{ds}\left(\dfrac{d\underline{r}}{dt}\dfrac{dt}{ds}\right) = \dfrac{d^2\underline{r}}{dt^2}\left(\dfrac{dt}{ds}\right)^2 = \dfrac{1}{v^2}\dfrac{d^2\underline{r}}{dt^2}$

We define the instantaneous radius vector of the particle motion as $\underline{R_c} = -\dfrac{\underline{\hat{n}}}{|\underline{r}''|}$

so that $\underline{r}'' = \dfrac{\underline{\hat{n}}}{R_c} = \dfrac{-\underline{R_c}}{R_c^2}$

The centrifugal force $\underline{F_{cf}}$ acting on the particle directed along $\underline{R_c}$:

$$\underline{F_{cf}} = -m\frac{d^2\underline{r}}{dt^2} = -mv^2\underline{r}'' = -mv^2\frac{\underline{\hat{n}}}{R_c} = mv^2\frac{1}{R_c}|\underline{r}''|\,\underline{R_c}$$

$$= m\frac{v^2}{R_c^2}\underline{R_c}$$

Applied to a curved magnetic field and derivation of the curvature drift.

Following on from the general analysis of the motion of a particle along a curved path, if the particle is moving in a curved magnetic field, we can determine the centrifugal force acting on the particle and hence deduce the curvature drift motion.

The unit vector tangent to a magnetic field line is $\frac{\underline{B}}{B}$

so that we can identify this with the previous notation as $\underline{r}' = \frac{\underline{B}}{B}$ and $\underline{r}'' = \frac{\hat{\underline{n}}}{R_c} = \frac{d\underline{r}'}{ds}$

where s is the distance parameter measured along the magnetic field line, so that $\underline{r}'' = \frac{d}{ds}\left(\frac{\underline{B}}{B}\right)$

The gradient operator along the magnetic field line is given by $\frac{d}{ds} = \frac{\underline{B}}{B}\cdot\nabla$

(in other words, the projection of the gradient operator ∇ on the unit vector $\frac{\underline{B}}{B}$ along the field line).

This gives
$$\frac{\hat{\underline{n}}}{R_c} = \underline{r}'' = \frac{d}{ds}\left(\frac{\underline{B}}{B}\right) = \left(\frac{\underline{B}}{B}\cdot\nabla\right)\frac{\underline{B}}{B}$$

and the centrifugal force acting on the particle is
$$\underline{F}_{cf} = -mv^2\frac{\hat{\underline{n}}}{R_c} = -mv^2\left(\frac{\underline{B}}{B}\cdot\nabla\right)\frac{\underline{B}}{B}$$

More correctly, it is the velocity component of the particle parallel to the magnetic field, $\underline{v}_{\parallel}$ that needs to be used in the expression for the centrifugal force:
$$\underline{F}_{cf} = -mv_{\parallel}^2\left(\frac{\underline{B}}{B}\cdot\nabla\right)\frac{\underline{B}}{B} = mv_{\parallel}^2\frac{\underline{R}_c}{R_c^2}$$

Given the general formula for the drift velocity of a particle for a force $\underline{F}$:
$$\underline{V}_{drift} = \frac{\underline{F}\times\underline{B}}{qB^2}$$

We get the expression for the curvature drift
$$\underline{v}_{CD} = \frac{mv_{\parallel}^2}{qB}\frac{\underline{B}}{B}\times\left[\left(\frac{\underline{B}}{B}\cdot\nabla\right)\frac{\underline{B}}{B}\right] = \frac{mv_{\parallel}^2}{qB}\hat{\underline{b}}\times\left[\left(\hat{\underline{b}}\cdot\nabla\right)\hat{\underline{b}}\right]$$
$$= \frac{mv_{\parallel}^2}{q}\frac{\underline{R}_c\times\underline{B}}{R_c^2B^2}$$

here $\hat{\underline{b}} = \frac{\underline{B}}{B} = \underline{r}'$ is the unit vector along the magnetic field line.