Space Physics Problem Sheet 4

Question 1:
In lectures we derived the differential equation which governs the outflow of an isothermal gas (T=constant) solar wind, and showed that it described a transition from subsonic to supersonic flow at a critical radius. A more realistic assumption is that the gas cools adiabatically as it expands outwards. In this problem we will show that the equivalent differential equation in this case has the same property.

As before the governing equation of motion is:

$$v \frac{dv}{dr} = -\frac{1}{\rho}\frac{dp}{dr} - \frac{GM_s}{r^2}$$

And conservation of mass is $I = 4\pi r^2 \rho v$, where the total mass flux I is a constant. Rather than T = constant though we now have $p = K\rho^\gamma$, where γ is the ratio of specific heats (5/3 for a monatomic gas). Using the constancy of I and K, derive an expression for p in terms of r and v (and the constants), and differentiate this with respect to r to show that:

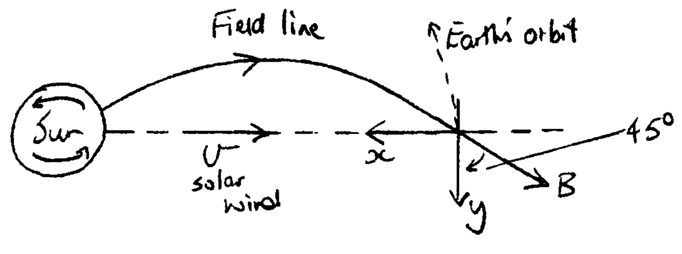
$$\frac{dp}{dr} = -\gamma p \left[\frac{2}{r} + \frac{1}{v}\frac{dv}{dr}\right]$$

Substitute this into the equation of motion to show that the governing differential equation is

$$\frac{dv}{dr} = \frac{v}{r}\frac{(2c_s^2 - GM_s/r)}{(v^2 - c_s^2)}$$

Where the sound speed for adiabatic flow is $c_s = (\gamma p/\rho)^{1/2}$. Again the "solar wind" solution of this equation describes a transition from subsonic ($v < c_s$) to supersonic ($v > c_s$) flow across a critical radius given by $2c_{s(r=r_c)}^2 = \frac{GM_s}{r_c}$.

Question 2:
At the orbit of the Earth, the average magnetic field in the solar wind (the interplanetary magnetic field) lies in the ecliptic plane (the plane of the Earth's orbit) at 45° to the Earth-Sun line, as shown in the sketch. The average strength of the field is 7 nT. If the solar wind flows radially outwards from the Sun at 500 kms^{-1}, calculate the magnitude and direction of the interplanetary electric field.



Question 3:

3a) A simple model of the combined corotation/solar wind-driven Dungey cycle flow may be obtained by taking the electrostatic potential in the equatorial plane to be (as in lectures)

$$\Phi(r,\phi) = -\left[E_0 r \sin\phi + \frac{\omega_p B_{eq} R_p^3}{r}\right]$$

where Eo is the cross-magnetosphere electric field associated with the Dungey cycle, ω_p is the angular frequency of the planetary rotation, B_{eq} is the equatorial field strength at the planet's surface; R_p the planet's radius and azimuthal angle ϕ is measured positive anticlockwise from the noon meridian towards dusk. The gradient vector in polar coordinates in a plane is

$$\underline{\nabla} = \frac{\partial}{\partial r}\hat{\underline{r}} + \frac{1}{r}\frac{\partial}{\partial\phi}\hat{\underline{\phi}}$$

Use this to find the position of the equatorial stagnation point (R_{sp}, ϕ_{sp}) in the flow (i.e. where $\underline{E} = -\nabla\Phi = 0$)

Show that the value of the potential at the stagnation point is $\Phi_{sp} = -2E_0 R_{sp}$.

3b) The location of the stagnation streamline, the streamline which passes through the stagnation point and which divides up the flow regions in the equatorial plane, can be found by writing $\Phi(r,\phi) = \Phi_{sp}$. Show that this equation reduces to the quadratic for (r/R_{sp})

$$(r/R_{sp})^2 \sin\phi - 2(r/R_{sp}) + 1 = 0$$

Show that the solution to this equation is

$$(r/R_{sp}) = \left(1 \pm \sqrt{1-\sin\phi}\right) / \sin\phi$$

Investigate both roots of this equation for $\phi = \pm 45°, \pm 90°, 0°$ (a limit will have to be taken in the latter case). Use these values and the mirror symmetry about the dawn-dusk meridian ($\phi = 90°, 270°$) to sketch the form of the stagnation streamline.

Question 4:

In this question we deal with the Earth's dipole magnetic field, some details of which can be seen in Figures 1 and 2 below. It can be seen that the magnetic dipole vector of the Earth points south, and hence the magnetic field in the equatorial plane is directed northward.

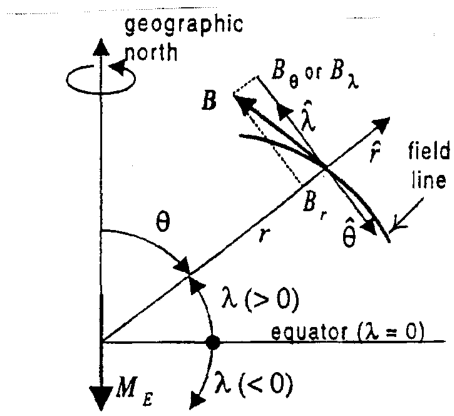


Fig. 1 Definition of the coordinate system for describing the Earth's dipole magnetic field (for simplicity, the axis of the dipole is shown to be coincident with the Earth's rotation axis).

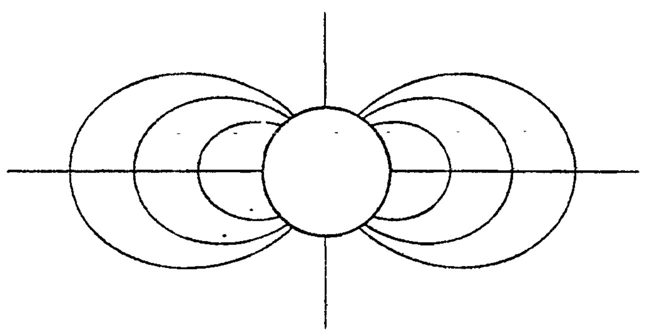
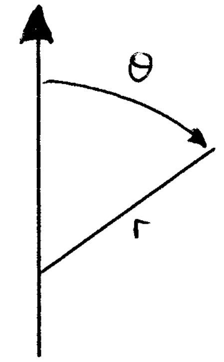


Fig. 2 Schematic diagram of the Earth's magnetic field lines. Field lines are shown for equatorial distances of $2R_E$, $3R_E$ and $4R_E$ where R_E is the radius of the Earth.

What now follows is a Past Exam question:



In the spherical co-ordinate system defined in the diagram above (in which the azimuth angle ϕ completes the right handed system) the Earth's magnetic field is given by its components:

$$B_r = \frac{-2M_E\cos\theta}{r^3}\ ,\qquad B_\theta = \frac{-M_E\sin\theta}{r^3}\ ,\qquad B_\phi = 0$$

where M_E is the magnitude of the Earth's dipole moment.

4a) Show that the magnitude of the magnetic field B is given by

$$B(r,\theta) = B_{eq}\frac{R_E^3}{r^3}\sqrt{1+3\cos^2\theta}$$

where R_E is the Earth's radius. What is B_{eq} ?

4b) Derive in general the differential equation of a magnetic field line in spherical polar coordinates. What is the differential equation which defines a magnetic field line in the Earth's field? Integrate the differential equation found above for the Earth's field to show that the equation of a field line is

$$r = r_{eq}\sin^2\theta$$

What is the meaning of r_{eq} ?

4c) If we label a magnetic field line by the parameter $L = r_{eq}/R_E$ show that the magnetic field strength along a field line is

$$B(L,\theta) = \frac{B_{eq}}{L^3}\frac{\sqrt{3\cos^2\theta+1}}{\sin^6\theta}$$

4d) Find the expression for the polar angle θ_L of the intersection of the magnetic field line, defined by the parameter L, with the Earth's surface.

(End of past exam question – what follows are 2 additional questions on the same topic)

And in addition:

4e) Given that B_{eq} = 30,000 nT and R_E = 6,400 km, calculate the magnitude of the magnetic field line in the equatorial plane: (i) at a distance of 10R_E , (ii) at the orbit of geostationary satellites, r = 6.6 R_E, (iii) at 2.5 R_E, close to the location of the peak intensity of the trapped (Van Allen) radiation belt, and (iv) at a height of 200km, in the ionosphere.

4f) In the geomagnetic field, the motion of plasma and energetic particles depends on the geometry of the field. The geometry, and therefore the particle motion depends on differential vector functions of the dipole field. The following is one example of such an expression.

Calculate the expression $\underline{B} \times \nabla \underline{B}$ at the equator $\left(\theta = \frac{\pi}{2}\right)$ and sketch the direction of the three vectors

$$\underline{B}, \; \nabla\underline{B}, \; \text{and} \; \underline{B} \times \nabla\underline{B}.$$

Use as necessary the expression for the gradient of the scalar quantity $B(r,\theta)$ in spherical polar co-ordinates:

$$\nabla B_r = \frac{\partial B}{\partial r} \quad \text{and} \quad (\nabla B)_\theta = \frac{1}{r}\frac{\partial B}{\partial \theta}$$

Without further calculation, indicate the direction of the vector $\underline{B} \times \nabla \underline{B}$ away from the equatorial plane (i.e. when $\theta \neq \pi/2$).

Question 5:

Charged particles moving in a magnetic field $\underline{B}$ which varies in strength in a direction perpendicular to $\underline{B}$ acquire a drift velocity $\underline{v}_{gd}$ given by:

$$\underline{v}_{gd} = \frac{W_\perp}{qB^3} \underline{B} \times \nabla \underline{B}$$

Where $W_\perp$ is the kinetic energy of the particles corresponding to the component of their velocity perpendicular to the magnetic field, q is the electric charge of the particle (positive for positively charged particles and negative for electrons) and $B = |\underline{B}|$ is the magnitude of the magnetic field.

5a) In the Earth's magnetic equatorial plane, the magnetic field at a distance r from the centre of the Earth is given as

$$\underline{B}_{eq}(r) = B_0 \left(\frac{R_E}{r}\right)^3 \hat{\underline{z}}$$

where B_0 = 31,000 nT (the field strength at the surface of the Earth at the magnetic equator), R_E is the radius of the earth and $\hat{\underline{z}}$ is the unit vector pointing north. Show that

$$\left(\underline{B} \times \nabla\underline{B}\right) = -3 \frac{[B_{eq}(r)]^2}{r} \hat{\underline{\phi}}$$

where $\hat{\underline{\phi}}$ is the unit vector in the azimuthal direction, pointing east.

5b) Show that the charged particles trapped in the equatorial plane of the Earth's magnetic field (with a pitch angle – the angle between the magnetic field vector and the particle's velocity vector of 90°) undergo a gradient drift motion in the Earth's dipole field, and that electrons and positive ions drift in opposite directions around the Earth. Comment on the sense of the resultant current and the magnetic field generated by it with respect to the Earth's field.

5c) Calculate (i) the magnetic field $B_{eq}(r)$ for r = 7 R_E, (ii) the drift velocity of positive ions with 90° equatorial pitch angle of energy 100 keV and 10 MeV and (iii) the time taken by these positive ions to drift around the Earth (i.e. their drift periods).