Space Physics Problem Sheet 4 - Solutions

Question 1:

Using $\rho = \left(\frac{I}{4\pi r^2 v}\right)$, we have $p = k\rho^\gamma = k\left(\frac{I}{4\pi r^2 v}\right)^\gamma$,

and differentiating this with respect to r, we find:

$$\frac{dp}{dr} = -\gamma k\left(\frac{I}{4\pi r^2 v}\right)^\gamma \left[\frac{2}{r} + \frac{1}{v}\frac{dv}{dr}\right]$$

$$= -\gamma p\left[\frac{2}{r} + \frac{1}{v}\frac{dv}{dr}\right]$$

Substituting this into equation of motion and collecting terms in $\frac{dv}{dr}$, we hence find:

$$\frac{dv}{dr} = \frac{v}{r}\frac{(2C_s^2 - GM_s/r)}{(v^2 - C_s^2)}$$

Question 2:

Using the Cartesian co-ordinate system shown in the sketch, we have

$\underline{B} = \left(-\frac{B}{\sqrt{2}}, +\frac{B}{\sqrt{2}}, 0\right)$ where $B = 7\,nT$, and

$\underline{v} = (-v, 0, 0)$ where $v = 500\,kms^{-1}$.

Thus $\underline{E} = -\underline{v}\times\underline{B} = \left(\frac{vB}{\sqrt{2}}\right)\hat{z}$. The magnitude of the electric field is therefore:

$$E = \frac{500\times10^3 \times 7\times10^{-9}}{\sqrt{2}}\ Vm^{-1} = 2.5\times10^{-3}\ Vm^{-1}$$
$$= 2.5\ mV\,m^{-1}$$

and it points north, perpendicular to the ecliptic.

Question 3:

3a) The electric field is given by:

$$\underline{E} = \frac{\partial}{\partial r}\left[E_0 r \sin\phi + \frac{W_p B_{eq} R_p^3}{r}\right]\hat{\underline{r}} + \frac{1}{r}\frac{\partial}{\partial \phi}\left[E_0 r \sin\phi + \frac{W_p B_{eq} R_p^3}{r}\right]\hat{\underline{\phi}}$$

$$= \left[E_0 \sin\phi - \frac{W_p B_{eq} R_p^3}{r^2}\right]\hat{\underline{r}} + E_0 \cos\phi\, \hat{\underline{\phi}} = 0$$

The phi component is zero when $\phi = 90°$ or $270°$ (dusk or dawn meridians), but if E_0 is positive then the radial component can only be zero if $\sin\phi$ is positive (so the dusk meridian, where $\theta = 90°$). The radial distance at which the radial component is then zero, is given by:

$$R_{sp} = \sqrt{\frac{W_p B_{eq} R_p^3}{E_0}}$$. This condition also implies

$$E_0 R_{sp} = \frac{W_p B_{eq} R_p^3}{R_{sp}}$$, and using this and $\phi_{sp} = 90°$

in the formula for $\Phi(r,\phi)$ yields:

$$\Phi_{sp} = -2E_0 R_{sp} = -2\left(\frac{W_p B_{eq} R_p^3}{R_{sp}}\right)$$

3b) The formula for the stagnation streamline is

$$\Phi(r,\phi) = -\left[E_0 r\sin\phi + \frac{W_p B_{eq} R_p^3}{r}\right] = -\left[E_0 r\sin\phi + E_0\frac{R_{sp}^2}{r}\right] = -2E_0 R_{sp}.$$

Thus multiplying by r and dividing by $E_0 R_{sp}^2$ yields the quadratic

$$\left(\frac{r}{R_{sp}}\right)^2 \sin\phi - 2\left(r/R_{sp}\right) + 1 = 0$$

with solutions: $$\left(r/R_{sp}\right) = \frac{\left(1 \pm \sqrt{1-\sin\phi}\right)}{\sin\phi}$$

For $\phi = 45°$; we have $(r/R_{sp}) = 2.18$ (+ sign), 0.65 (− sign), which correspond to the outer and inner branches of the stagnation streamline. (Note that by symmetry, the same values are obtained for $\phi = 135°$).

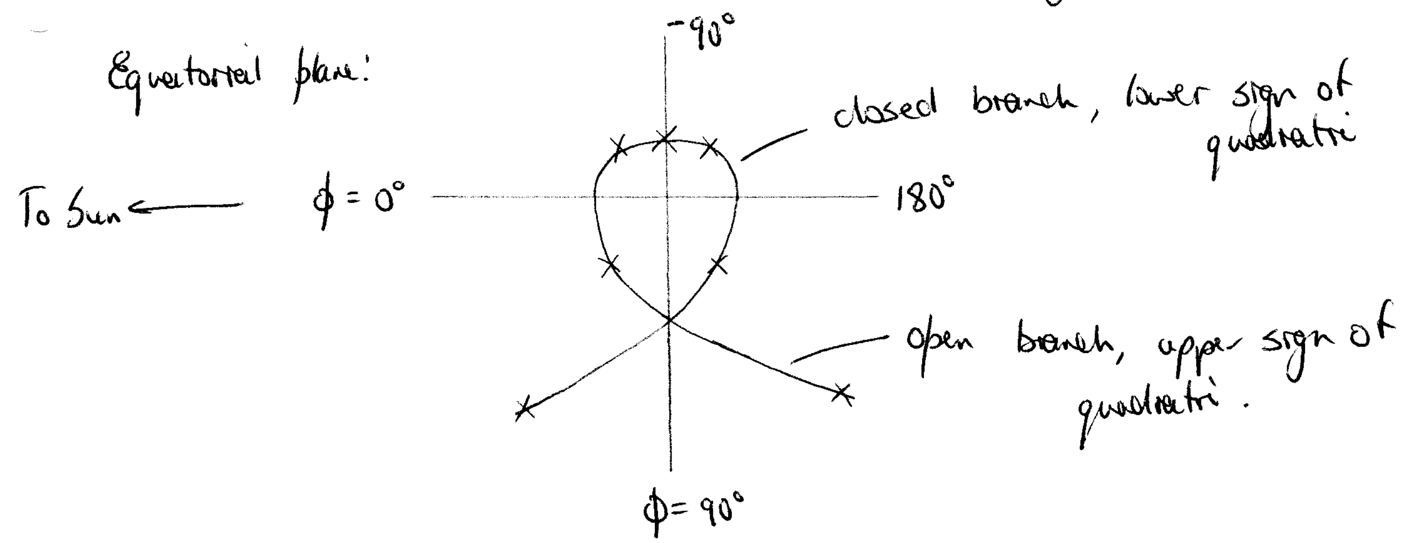
For $\phi = -45°$, we have $(r/R_{sp}) = -3.26$ (+ sign) and 0.43 (− sign). The first solution is unphysical since r must be a positive quantity, so the outer branch does not exist at this azimuth. (Same solution holds for $\phi = -135°$).

For $\phi = 90°$, we have $(r/R_{sp}) = 1$, which is the stagnation point of course, and for $\phi = -90°$, we have $(r/R_{sp}) = -2.41$ (+ sign) and unphysical, and 0.41 (− sign).

For $\phi = 0°$, we have $\sin\phi = 0$, so we need to take this limit in the formula:
This goes $(r/R_{sp}) = \left(1 \pm \left(1 - \frac{1}{2}\sin\phi\right)\right)/\sin\phi$, so $+$ sign goes $(r/R_{sp}) = 2/\sin\phi$, which goes to infinity (outer branch of stagnation streamline goes to infinity in this limit), while $-$ sign goes $(r/R_{sp}) = 0.5$ (Same solution as obtained for $\phi = 180°$).

On using these values, we can sketch the stagnation streamline:

Equatorial plane:



Question 4:

4a) $$B(r,\theta) = \sqrt{B_r^2 + B_\theta^2} = \frac{M_E}{r^3}\sqrt{4\cos^2\theta + \sin^2\theta}$$

$$= \frac{M_E}{r^3}\sqrt{1+3\cos^2\theta}$$

If we define B_{eq}, as the field strength at the Earth's surface, in the equatorial plane, then $r = R_E$, $\theta = 90°$,

hence $$B(r=R_E,\ \theta=90°) = B_{eq} = \frac{M_E}{R_E^3}$$

Hence $$B(r,\theta) = B_{eq}\frac{R_E^3}{r^3}\sqrt{1+3\cos^2\theta}$$

4b) The differential equation which defines a field line is:

$$\underline{dr} \times \underline{B} = 0$$, which in spherical polar co-ordinates can be written as:

$$\frac{dr}{B_r} = \frac{r\,d\theta}{B_\theta}$$

(where in the Earth's field, which is axisymmetric as given in this example, $B_\phi = 0$).

On re-arranging and substituting for B_r, B_θ, we get

$$\frac{dr}{r} = 2\frac{\cos\theta\,d\theta}{\sin\theta} = 2\frac{d(\sin\theta)}{\sin\theta}$$, which

is then integrated to give: $\ln r$ = constant + $2\ln(\sin\theta)$, or

r = constant $\times \sin^2\theta$

The magnetic field line crosses the equator for $\theta = 90°$, when $r = r_{eq}$, which is the integration constant, so the equation for the magnetic field line is:

$$r = r_{eq}\sin^2\theta.$$

4c). If we set $L = r_{eq}/R_E$, then the equation of the magnetic field line can be written as:

$$r = L R_E \sin^2\theta .$$

We substitute this expression (as it defines a field line) into the expression found in 4a) to get,

$$B(L,\theta) = B_{eq}\frac{R_E^3}{r^3}\sqrt{1+3\cos^2\theta} = \frac{B_{eq}}{L^3}\frac{\sqrt{1+3\cos^2\theta}}{\sin^6\theta} .$$

4d) Using the form of the equation of the field line from 4c), $r = L R_E \sin^2\theta$, we set $r = R_E$ (as the distance of interception from the origin is the radius of the Earth), to get

$$R_E = L R_E \sin^2\theta_c, \quad \text{so that} \quad \sin^2\theta_c = \frac{1}{L} .$$

4e) In the equatorial plane, $\theta = \pi/2 = 90°$, so that $B(r, \theta = 90°) = B_{eq}(R_E/r)^3$.

(i) For $r = 10 R_E$, $B = \frac{30{,}000}{10^3} = 30\,nT$

(ii) For $r = 6.6 R_E$, $B = \frac{30{,}000}{(6.6)^3} = 104\,nT$

(iii) For $r = 2.5 R_E$, $B = \frac{30{,}000}{(2.5)^3} = 1{,}920\,nT$

(iv) at a height of 200 km, $r = R_E + 200\,km = 6{,}600\,km$,

$$\text{so } B = (30{,}000)\left(\frac{6{,}400}{6{,}600}\right)^3 = 27{,}354\,nT.$$

4f) At the equator we have $\theta = 90°$, so that $B_r = 0$ and $B_\theta = -B_{eq}\left(\frac{R_E}{r}\right)^3$, the magnitude of the magnetic field is

$$B(r, \theta = 90°) = B_{eq}\frac{R_E^3}{r^3} .$$

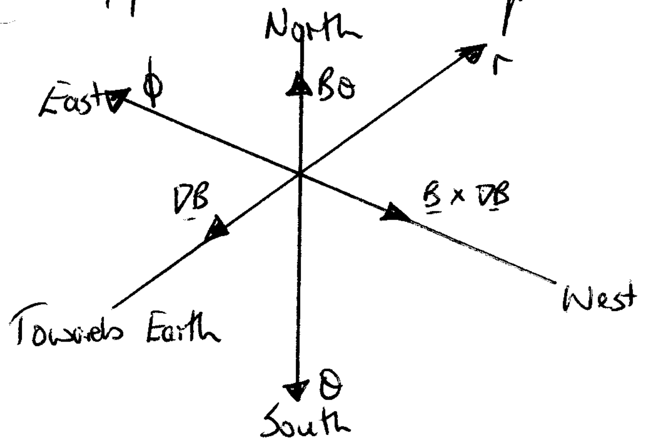
This means that:

$$(\nabla B)_r = \frac{\partial B}{\partial r} = -3B_{eq}\frac{R_E^3}{r^4}, \quad \text{and} \quad (\nabla B)_\theta = 0.$$

The cross-product will only have a non-zero component in the ϕ direction:

$$(\underline{B}\times\nabla\underline{B})_r = 0, \quad (\underline{B}\times\nabla\underline{B})_\theta = 0, \quad (\underline{B}\times\nabla\underline{B})_\phi = -B_\theta(\nabla B)_r$$
$$= -3B_{eq}^2\left(\frac{R_E}{r}\right)^3\frac{R_E^3}{r^4}$$
$$= -\frac{3B^2}{r}.$$

The directions of the vector are sketched below (valid while the r, ϕ axes are in equatorial plane)

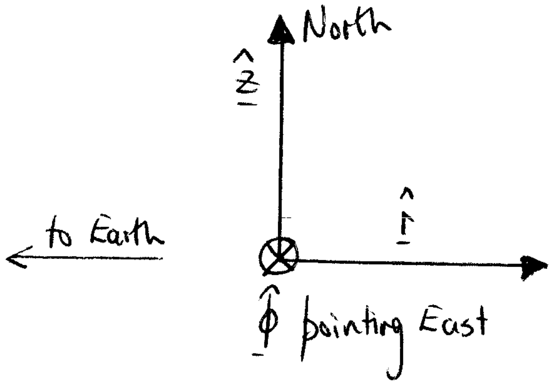


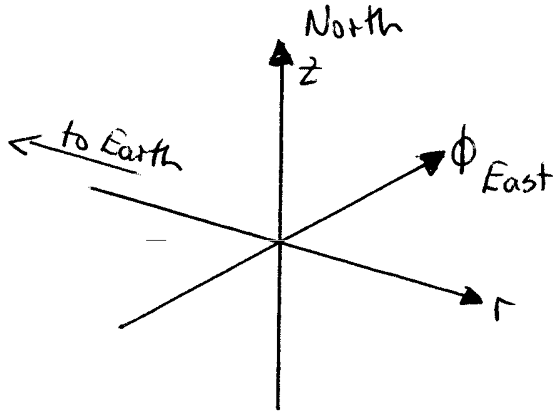
Away from the equatorial plane, the magnetic field vector still points towards the north pole along the tangent to the field line, the gradient vector of the field magnitude points towards the Earth in the meridian plane and therefore the cross product will still point westward, as in the equator.

Question 5:

5a) In this problem, it is particularly important to define the co-ordinate axes properly, to make sure we get the right results. In the first place, the magnetic field is normally given for a dipole, as it was in lectures, in a spherical co-ordinate system (and as in 4A).

Here we are dealing with the magnetic field in the Earth's magnetic equatorial plane, and so a cylindrical system can be used. This is why the magnetic field is given with a unit vector $\hat{z}$ pointing North. The co-ordinate axes are fully defined on next page:





In this co-ordinate system, the grad of a scalar function f is defined as:

$$\underline{\nabla} f = \hat{\underline{r}} \frac{\partial f}{\partial r} + \hat{\underline{\phi}} \frac{1}{r} \frac{\partial f}{\partial \phi} + \hat{\underline{z}} \frac{\partial f}{\partial z} .$$

In the case of the Earth's magnetic field in the equatorial plane,

$$\underline{B}_{eq}(r) = B_0 \left(\frac{R_E}{r}\right)^3 \hat{\underline{z}}$$

, so the field magnitude is

$B_{eq}(r) = B_0 \left(\frac{R_E}{r}\right)^3$. The magnitude of the field only depends on r, so applying the formula for the grad. we get:

$$\underline{\nabla} B_{eq}(r) = -\frac{3B_0}{r}\left(\frac{R_E}{r}\right)^3 \hat{\underline{r}} = -\frac{3}{r}\left[B_{eq}(r)\right] \hat{\underline{r}} .$$

Forming the cross-product, we get:

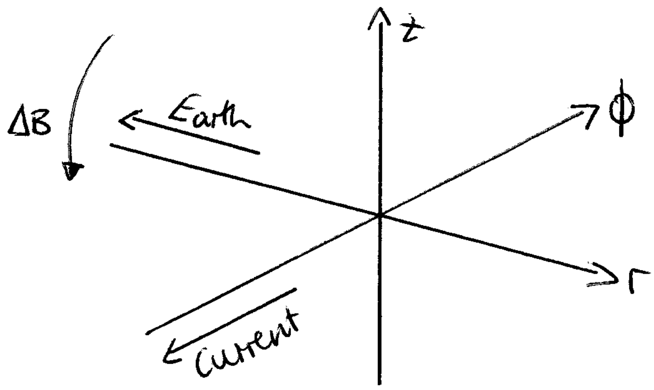
$$\underline{B} \times \underline{\nabla} B = -\frac{3}{r}\left[B_{eq}(r)\right]^2 \hat{\underline{z}} \times \hat{\underline{r}} = -\frac{3}{r}\left[B_{eq}(r)\right]^2 \hat{\underline{\phi}}$$

5b) Particles will undergo a grad. drift motion with a velocity:

$$\underline{v}_{gd} = \frac{W_\perp}{qB^3} \underline{B} \times \underline{\nabla} B = -\frac{W_\perp}{q\left[B_{eq}(r)\right]^3} \frac{3\left[B_{eq}(r)\right]^2}{r} \hat{\underline{\phi}}$$

$$= \frac{-3W_\perp}{qr\, B_{eq}(r)} \hat{\underline{\phi}}$$

Particles with a positive charge will move in the direction $-\hat{\underline{\phi}}$, i.e., westwards; electrons with their negative charge, will move in the direction of $+\hat{\phi}$, ie eastwards. The resultant current flows westwards around the Earth.



The magnetic field generated by this ring current around the Earth, ΔB (using the right hand rule around the current), is a southward pointing magnetic field, which is in the opposite direction to that of the Earth's dipole field.

An intensification of the ring current, which occurs at times of large solar storms hitting the magnetosphere, causes a depression in the Earth's field and is frequently observed. Particularly intense events can cause long-lasting depressions, called magnetic storms.

5c) (i) $B_{eq}(7R_E) = 90\ nT$

(ii) $v_{gd}(W = 100\ keV) = 74\ km s^{-1}$, $v_{gd}(W = 10\ MeV) = 7400\ km s^{-1}$

(iii) $T_{gd}(W = 100\ keV) = \dfrac{2\pi \times 7R_E}{v_{gd}} = 3803\ s.$

$T_{gd}(W = 10\ MeV) = 38s.$