Space Physics – Solutions 1

Question 1

a)We have :

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right)=\frac{-en_o}{\varepsilon_o}\left[1-\exp\left(\frac{e\Phi(r)}{kT}\right)\right]$$

and if kT >> $e\Phi(r)$, then we can expand the exponential function in a Taylor series ;

that is, for x<<1 , $e^x = 1 + x + \frac{x^2}{2!} + \ldots\ldots$

if we let $x = \frac{e\Phi(r)}{kT}$,

then

$$\exp(x) = \exp\left(\frac{e\Phi(r)}{kT}\right) \simeq 1 + \frac{e\Phi(r)}{kT}$$, and then

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right) \simeq \frac{-en_0}{\varepsilon_0}\left[1-1-\frac{e\Phi(r)}{kT}\right] = \frac{n_0e^2}{\varepsilon_0 kT}\Phi(r)$$

$$= \frac{\Phi(r)}{\lambda_D}$$, when λ_D = Debye length = $\sqrt{\frac{\varepsilon_0 kT}{n_0 e^2}}$

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b) The differential equation is:

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\Phi(r)}{dr}\right)=\frac{\Phi(r)}{\lambda_D^2}$$

and the suggested solution is given by:

$$\Phi(r)=\frac{A}{r}\exp\left(\frac{-r}{\lambda_D}\right)$$

If we differentiate the solution with respect to r then,

$$\frac{d\Phi(r)}{dr} = -\frac{A}{r^2}\exp\left(\frac{-r}{\lambda_D}\right) - \frac{A}{\lambda_D r}\exp\left(\frac{-r}{\lambda_D}\right) = -\frac{A}{r}\exp\left(\frac{-r}{\lambda_D}\right)\left[\frac{1}{r}+\frac{1}{\lambda_D}\right]$$

Hence on substitution into the left hand side of the differential equation we obtain:

$$\frac{1}{r^2}\frac{d}{dr}\left[-rA\exp\left(\frac{-r}{\lambda_D}\right)\left(\frac{1}{r}+\frac{1}{\lambda_D}\right)\right] = \frac{1}{r^2}\frac{d}{dr}\left[-A\exp\left(\frac{-r}{\lambda_D}\right) - \frac{r}{\lambda_D}A\exp\left(\frac{-r}{\lambda_D}\right)\right]$$

$$= \frac{1}{r^2}\left[\cancel{\frac{A}{\lambda_D}\exp\left(\frac{-r}{\lambda_D}\right)} - \cancel{\frac{A}{\lambda_D}\exp\left(\frac{-r}{\lambda_D}\right)} + \frac{r}{\lambda_D^2}A\exp\left(\frac{-r}{\lambda_D}\right)\right]$$

$$= \frac{1}{r\lambda_D^2}A\exp\left(\frac{-r}{\lambda_D}\right)$$ = right hand side of differential equation (QED)

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c) Using the expression for the Debye length as derived in lectures,

$$\lambda_D = \sqrt{\frac{\varepsilon_0 k T}{n_0 e^2}}$$

and the numerical values given in the question, the required expression can be simply derived as shown:

$$\lambda_D = \sqrt{\frac{8.8542\times10^{-12} \times 1.3807\times10^{-23} \times T\times10^{6}}{n_0 \times (1.6022\times10^{-19})^2}} \qquad \underbrace{\sqrt{\frac{F m^{-1} J \; K^{0\,-1}\; K^{0}}{cm^{-3}\; C^2}}}_{*}$$

$$= 69\times10^{3}\sqrt{\frac{T}{n_0}}\;[\text{unit}]$$

For units: $F = \frac{C}{V} = \frac{A^2 s}{W} = \frac{A^2 s^2}{Nm} = \frac{A^2 s^2}{kg\, m^2 s^{-2}}$

$J = kg\, m^2 s^{-2}$, $C^2 = A^2 s^2$, $m^{-1} = 10^{-2}\, cm^{-1}$

Hence $*$ becomes $\sqrt{\frac{\cancel{A^2 s^2} m^{-1}\; \cancel{kg\, m^2 s^{-2}}}{\cancel{kg\, m^2 s^{-2}}\; cm^{-3}\; \cancel{A^2 s^2}}} = \frac{cm}{10}$

Hence $\lambda_D = 69\times10^{3}\sqrt{\frac{T}{n_0}}\left[\frac{cm}{10}\right] \equiv 69\sqrt{T/n_0}\;[m]$

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d) The Debye length for the ionosphere can be calculated to be:

$$\lambda_D = 0.7\,cm \approx 1\,cm$$

This Debye length is much less than either the vertical (L ~ 300km) or horizontal (L ~ 3,000km) extent of the ionosphere; hence $\lambda_D << L$, and so the ionosphere can be considered to be quasi-neutral.

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e) The Debye length of the solar wind can be calculated to be:

$$\lambda_D = 7m$$

Seven metres is much less than the macroscopic spatial scale of the solar wind (L ~ 1AU ~ 10^8km) and hence the solar wind can also be considered to be quasi-neutral. However note that λ_D is greater than, or comparable to, the size of most spacecraft that have traversed the interplanetary medium. This must be taken into account when designing instruments to measure solar wind properties.

Figure 1 shows the values of the Debye length for a range of typical parameters for a number of space plasma regions. One can see that in the inner magnetosphere, plasmasphere, of the Earth, that the Debye length is of the order of cm, whereas in the solar wind (as we have calculated) it is of the order of 10m and in the Earth's tail region 100's of metres. As we examine these differing plasmas in the course these numbers will allow us to decide whether the plasmas we are examining or quasi-neutral or not.

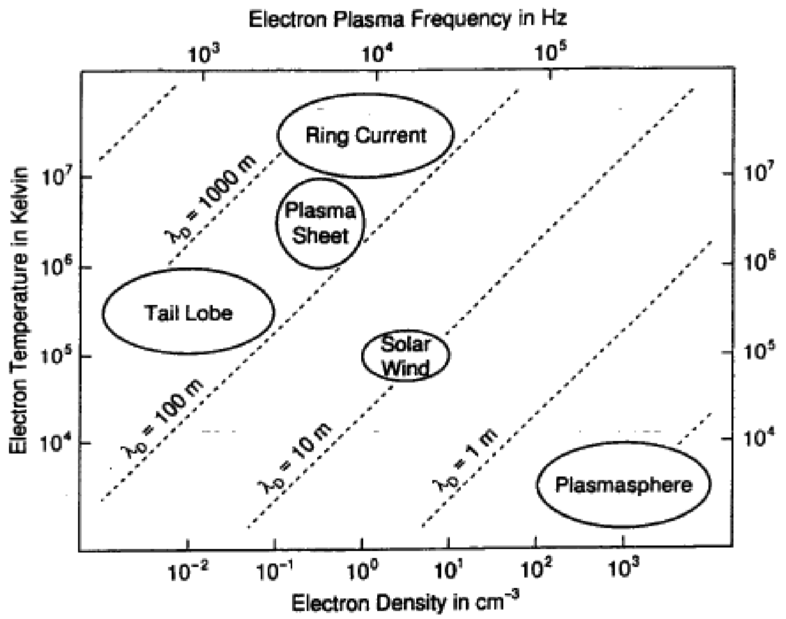

Figure 1 : Ranges of typical parameters for different types of space plasmas (taken from Baumjohann and Treumann, 1997).

Question 2:

a) We derived a formula in lectures for the electron plasma frequency, ω_{pe}, given by:

$$\omega_{pe}^2 = \frac{n_o e^2}{\varepsilon_o m_e}$$

Hence:

$$f_{pe}\,(Hz) = \frac{\omega_{pe}}{2\pi} = \frac{1}{2\pi}\sqrt{\frac{n_o e^2}{\varepsilon_o m_e}}$$

$$= \frac{1}{2\pi}\sqrt{\frac{n_o \times (1.6022 \times 10^{-19})^2}{8.8542 \times 10^{-12} \times 9.1095 \times 10^{-31}}} \left[\sqrt{\frac{cm^{-3}\,C^2}{Fm^{-1}\,kg}}\right]$$

$$= 8.98\sqrt{n_o} \times 10^3\ [Hz]\ ,\ \text{(check units as before!)}$$

$$\doteq 9 \times 10^3 \sqrt{n_o}\ [Hz]$$

b) The value of the electron plasma frequency shown in the data is

$$f_{pe} \doteq 6\ kHz = 6 \times 10^3\ Hz$$

Hence

$$n_o = \left(\frac{f_{pe}}{9 \times 10^3}\right)^2 = \left(\frac{6}{9}\right)^2 = 0.4\ cm^{-3}$$

Such oscillations are seen at all bow shocks, including that of the Earth. It is believed that these plasma oscillations are generated by an instability due to the existence of mildly relativistic electrons streaming away from the shock.

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c) We can write $\lambda_D^2 = \dfrac{\varepsilon_0 k T}{n_0 e^2}$

and $\omega_{pe}^2 = \dfrac{n_0 e^2}{\varepsilon_0 m_e}$

Hence $\lambda_D^2 \omega_{pe}^2 = \dfrac{\cancel{\varepsilon_0} k T}{\cancel{n_0 e^2}} \dfrac{\cancel{n_0 e^2}}{\cancel{\varepsilon_0} m_e}$

$$= \frac{kT}{m_e} = v_{th}^2$$

where v_{th} = electron thermal speed.

On checking the units:

$$\left[\frac{kT}{m_e}\right] = \left[\frac{\frac{J}{°K}\ °K}{kg}\right] = \frac{Nm}{kg} = \frac{kg\, m^2 s^{-2}}{kg}$$

$$= (ms^{-1})^2$$

$$= [v_{th}]^2$$