Space Physics Problem Sheet 2

Question 1:
A particle of mass m and charge q moves in a magnetic field $\underline{B} = (0, 0, B)$ constant in time, in which the strength of the field varies as a function of y: $B = B(y)$. We assume that the variation of B is small on the scale of the gyroradius r_L of the particle:

$$\frac{dB}{dy} << \frac{B}{r_L}$$

so that we can expand $B(y)$ into a Taylor series around the gyrocentre of the particle

$$B(y) = B_0 + (y - y_0)\frac{dB}{dy}$$

retaining only the first order term. Here B_0 is the approximate value of the magnetic field experienced by the particle in the gyration period which we examine.

1a) If the components of the velocity of the particle are v_x, v_y and v_z, show that the equations describing the motion of the particle are:

$$\frac{d^2 v_x}{dt^2} = -\left(\frac{qB}{m}\right)^2 v_x + v_y^2 \frac{q}{m}\frac{dB}{dy} \qquad (1)$$

$$\frac{d^2 v_y}{dt^2} = -\left(\frac{qB}{m}\right)^2 v_y - v_x v_y \frac{q}{m}\frac{dB}{dy} \qquad (2)$$

$$\frac{dv_z}{dt} = 0 \qquad (3)$$

Hint: to derive (1) and (2), note that the time derivative of B is that seen by the particle so that

$$\frac{dB}{dt} = \frac{dB}{dy}\frac{dy}{dt}$$

1b) As the variation of $\underline{B}$ is small over the gyroradius of the particle, substitute the Taylor expansion for $B(y)$ in (1) and (2), and note that the velocities, as well as $(y\text{-}y_0)$ can be replaced in the first order terms in dB/dy by the expressions derived in the course for a constant $\underline{B}$. Show that using these approximation equations (1) and (2) can be rewritten:

$$\frac{d^2 v_x}{dt^2} + \left(\frac{qB_0}{m}\right)^2 v_x = -\frac{q}{m}\frac{v_\perp^2}{2}\frac{dB}{dy}\left[1 - 3\cos(2\Omega t)\right] \qquad (4)$$

$$\frac{d^2 v_y}{dt^2} + \left(\frac{qB_0}{m}\right)^2 v_y = -3\frac{q}{m}\frac{v_\perp^2}{2}\frac{dB}{dy}\sin(2\Omega t) \qquad (5)$$

1c) Find the solutions of (4) and (5) in the form of a sum of two terms, the first being the gyromotion of the particle as derived for a constant $\underline{B}$ field, the second being a particular solution assuming that the d^2/dt^2 terms are zero.

1d) Use the solutions obtained in part (c) to calculate the displacement of the particle in the x and y directions during one gyroperiod, and show that its average velocity during a gyroperiod is

$$\langle v_x \rangle = -\frac{mv_\perp^2}{2}\frac{1}{q}\frac{1}{B^2}\frac{dB}{dy}$$

$$\langle v_y \rangle = 0$$

1e) Show that this result is a special case of the general form of the gradient drift velocity

$$v_{grad} = \frac{mv_\perp^2}{2qB^3}\left(\underline{B} \times \nabla \underline{B}\right) \qquad (6)$$

1f) Given that the gradient drift given by (6) arises as a result of a force $\underline{F}_{grad}$ acting on the particle, find the expression for it.

Question 2:

In this question we explore the motion of a charged particle in a magnetic field which is mostly along the z axis, but has either a changing magnitude (i.e. a spatial gradient) along z, or that it depends on time (i.e. it has a non-zero time derivative, which implies the existence of an electric field, according to Faraday's law). In both cases the main component of the particle's motion remains gyration around the magnetic field (as though B were constant along z), therefore around the z axis. This is because we assume that the spatial variation in the magnetic field is small on the scale of the gyroradius $r_L = mv_\perp/qB$ of the particle and the time variation in the magnetic field is slow on the scale of the gyroperiod $T_L = 2\pi/\Omega = 2\pi m/qB$. However its motion is modified in some small but important ways; at the same time, a quantity associated with the particle's motion, i.e. its magnetic moment (defined below) remains constant.

2a) A particle's magnetic moment is defined as

$$\mu = \frac{1}{2}\frac{mv_\perp^2}{B}$$

where m is the mass of the particle, $v_\perp$ is its velocity perpendicular to the magnetic field and B is the strength of the magnetic field. Show that this is equivalent to the definition of the magnetic moment of a current loop I enclosing an area A.

2b) We consider the motion of a particle in a magnetic field which is (mostly) along the z axis and is cylindrically symmetric around the z axis, but with a strength which is changing along z. The component of the field along z is B_z, and the field strength is $B \approx B_z$, but it has a non-zero gradient $\partial B_z/\partial z \approx \partial B/\partial z$ along z which can be taken to be a constant on the scale of the gyromotion of the particle. Maxwell's equation $\nabla\cdot\underline{B}=0$ in cylindrical co-ordinates is written as

$$\frac{1}{r}\frac{\partial}{\partial r}\left(rB_r\right) + \frac{1}{r}\frac{\partial B_\varphi}{\partial \varphi} + \frac{\partial B_z}{\partial z} = 0$$

Calculate the component of $\underline{B}$ perpendicular to z which is needed to satisfy the $\nabla\cdot\underline{B}=0$ condition.

2c) By noting that the additional force term in the equation of motion of the particle is along the z axis, and assuming that the gyromotion of the particle is not affected by any gradients in one gyroperiod, show that the force along z is

$$F_z = -\mu\frac{\partial B_z}{\partial z} \approx -\mu\frac{\partial B}{\partial z}$$

2d) Show that the rate of change of the particle's energy parallel to the magnetic field (i.e. its dominant component along the z axis) satisfies the following relationship

$$\frac{d}{dt}\left(\frac{1}{2}mv_\parallel^2\right) = -\mu\frac{dB}{dt}$$

where the time derivative is the rate of change of the magnetic field seen by the particle.

2e) Using the result from part d) and the fact that the total energy of the particle

$$\frac{1}{2}mv^2 = \frac{1}{2}mv_\perp^2 + \frac{1}{2}mv_\parallel^2$$

in a magnetic field remains constant, show that

$$\frac{d\mu}{dt} = 0$$

Note that the time derivative here represents the rate of change of the particle's magnetic moment along the trajectory of the particle.

2f) We now assume, in the remainder of this problem, that $\underline{B}$ remains parallel to the z axis, but is slowly variable in time; $\underline{B} = B(t)\, z$. Show that the rate of change in the kinetic energy of the particle perpendicular to $\underline{B}$, due to the electric field $\underline{E}$ induced by the time dependence of $\underline{B}$ is

$$\frac{d}{dt}\left(\frac{mv_\perp^2}{2}\right) = q\underline{E} \cdot \underline{v}_\perp$$

2g) Show that, if we assume that the amount of energy gain during one gyroperiod is small, so that the projection of the particle's orbit around $\underline{B}$ in the x-y plane is closed, the perpendicular energy gain is

$$\Delta\left(\frac{mv_\perp^2}{2}\right) = -q\oint \underline{E} \cdot \underline{dl}$$

where $\underline{dl} = \underline{v}_\perp\, dt$ is the infinitesimal path length along the particle's orbit and the integral is calculated over one orbit around the magnetic field. Explain the sign of the integral.

2h) Use Faraday's law to show that if the change in the magnetic field during one gyroperiod T_L is ΔB, with

$$\frac{\Delta B}{T_L} \approx \frac{\partial B}{\partial t}$$

then the amount of change in the energy of the particle, perpendicular to the magnetic field is

$$\Delta\left(\frac{mv_\perp^2}{2}\right) = \mu \Delta B$$

2i) Use the definition of μ to show that the result in 2h) implies that

$$\Delta\left(\frac{mv_\perp^2}{2B}\right) = \Delta\mu = 0$$

The results in question 2 imply that the magnetic moment μ of the particle remains at least approximately constant in magnetic fields which vary little in time and/or space on the scale of the gyromotion of the particle. This is why μ is called an adiabatic invariant of the particle's motion.