Space Physics : Problem Sheet 3

Question 1 (Past exam. Question)

1a) In a time-dependent magnetised plasma, the force balance equation is written as

$$-\nabla p + \underline{j} \times \underline{B} = 0$$

Define and comment briefly on the terms in this equation.

1b) A cylindrically symmetric plasma structure consists of a uniform current (along the *z* direction), confined to a radius *R*, with a total current given as *I*. Show that the magnetic field vector $\underline{B}(r, \varphi, z)$ generated by the current is purely azimuthal (i.e., $B_\varphi \neq 0$, $B_r = B_z = 0$).

1c) Find the magnetic field vector $\underline{B}(r, \varphi, z)$ inside and outside the cylinder as a function of *r* (the distance from the axis of the cylinder) and the total current *I*. (Use Ampere's law either in its differential or integral form).

1d) Use the force balance equation to find the expression for dp/dr for $r \leq R$. Show that if the plasma pressure outside the cylinder is constant (given as p_0), the plasma pressure inside the cylinder ($r \leq R$) is

$$p(r) = p_0 + \frac{\mu_0}{4}\left(\frac{I}{\pi R^2}\right)^2 (R^2 - r^2)$$

1e) Calculate the magnetic pressure outside the cylinder ($r > R$) and deduce, applying the force balance equation given in 1a), the magnetic tension force for ($r > R$).

The differential vector operator *curl* in cylindrical coordinates is as follows:

$$(\nabla \times \underline{B})_r = \frac{1}{r}\frac{\partial B_z}{\partial \phi} - \frac{\partial B_\phi}{\partial z}, \quad (\nabla \times \underline{B})_\phi = \frac{\partial B_r}{\partial z} - \frac{\partial B_z}{\partial r}, \quad (\nabla \times \underline{B})_z = \frac{1}{r}\frac{\partial}{\partial r}(r B_\phi) - \frac{1}{r}\frac{\partial B_r}{\partial \phi}$$

Question 2:

Ampere's law (if we ignore the contribution of the displacement current to the current density vector, an assumption usually made in space physics) is $\nabla \times \underline{B} = \mu_0 \underline{j}$ where $\underline{j}$ is the current density vector. In the absence of currents, the magnetic field satisfies $\nabla \times \underline{B} = 0$. Use this relation, to show that if the magnetic field vector is given in a cylindrical coordinate system as $\underline{B}(r) = B_\phi(r)\,\hat{\phi}$, with the other components $B_r = B_z = 0$ then $\frac{\partial B_\phi}{\partial r} + \frac{B_\phi}{r} = 0$.

Use as necessary the expression for the curl of an arbitrary vector $\underline{A} = A_r \hat{r} + A_\phi \hat{\phi} + A_z \hat{z}$ in cylindrical coordinates:

$$(\nabla \times \underline{A})_r = \frac{1}{r}\frac{\partial A_z}{\partial \phi} - \frac{\partial A_\phi}{\partial z}, \quad (\nabla \times \underline{A})_\phi = \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \quad (\nabla \times \underline{A})_z = \frac{1}{r}\frac{\partial}{\partial r}(r A_\phi) - \frac{1}{r}\frac{\partial A_r}{\partial \phi}$$

Question 3:

Given a scalar function $f(x, y, z)$ in Cartesian coordinates, verify, by calculating explicitly, the identity $\nabla \times (\nabla f) = 0$. Note that this is applied to the magnetic field in the absence of currents (e.g. question 2), when a scalar potential can be used to derive the magnetic field. There are many areas of application of this identity in space physics, one example is the modelling of magnetic fields in the corona.

Question 4:

A vector identity frequently used in space physics (because it is often used in fluid mechanics and plasma physics) is $\nabla \cdot (f\underline{A}) = f\,\nabla \cdot \underline{A} + \underline{A} \cdot \nabla f$, where f and $\underline{A}$ are arbitrary (respectively scalar and vector) functions. If $f(r, \theta, \phi)$ and $\underline{A} = A_r(r,\theta,\phi)\,\hat{r} + A_\theta(r,\theta,\phi)\,\hat{\theta} + A_\phi(r,\theta,\phi)\,\hat{\phi}$ are given in spherical polar coordinates, use the definitions given for grad and div below to evaluate the two sides of this identity and show that indeed they are equal. (This is really an exercise in manipulating vectors and differential equations, but familiarity with such manipulations is a great help at exam time). $(\nabla f)_r = \frac{\partial f}{\partial r}$, $(\nabla f)_\theta = \frac{1}{r}\frac{\partial f}{\partial \theta}$, $(\nabla f)_\phi = \frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}$

$$(\nabla \cdot \underline{A}) = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(A_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial \phi}$$

Question 5:
Given that at the orbit of the Earth (at 1AU = 150 million km) the average number density of the solar wind is 6.6 cm^{-3}, its velocity is 450 kms^{-1}, and assuming that the solar wind consists only of protons ($m_p = 1.67 \times 10^{-27}$ kg), calculate the rate (per second) of the mass outflow from the Sun in the form of the solar wind. Calculate the total mass of electrons the Sun also loses per second, to keep the solar wind a neutral plasma.

Question 6:
6a) The solar wind is a stream of plasma which is emitted uniformly at a constant radial speed $U_{SW} = 400\,kms^{-1}$ from the Sun. We assume in the following that the solar wind is given its constant radial velocity U_{SW} not directly on the solar surface, but on a sphere of radius $r_0 = 5R_S$ centred on the Sun (the source surface), where $R_S = 7\times10^5$ km is the solar radius. As the Sun rotates at an angular rate of $\omega_S = 2.86\times10^{-6}$ rads^{-1} the solar wind also acquires a transverse velocity component which corresponds to the circumferential velocity of the Sun at $r_0 = 5R_S$. Calculate this velocity. If the Earth is assumed to be in the solar equatorial plane, calculate the transverse velocity component of the solar wind at the distance of the Earth at $r_E = 1AU = 1.5\times10^8$ km due to this corotating velocity on the source surface (use the conservation of angular momentum). Compare this transverse velocity at the orbit of the Earth to U_{SW} ; can it be neglected?

6b) The Earth is orbiting the Sun in the sense of the solar rotation with a period of 1 year = 365 days. Calculate the apparent transverse velocity component of the solar wind (which flows radially from the Sun in an inertial reference system, when we ignore the transverse velocity component calculated in 6a)) detected by an observer at the orbit of the Earth. Calculate the apparent direction of the solar wind flow for an Earth based observer.

6c) In the frame of reference rotating with the Sun, the solar wind is emitted radially as well as uniformly. Show that in this frame the solar wind has also a transverse velocity, dependent on the distance *r* from the Sun. Calculate this transverse velocity $V_\phi(r)$ as a function of *r* (the distance from the centre of the Sun, ignoring again the transverse velocity arising from the effect described in 6a)) in the solar equatorial plane. Sketch the total velocity of the solar wind in this frame and show that the solar wind streamlines follow a spiral path in the equatorial plane.

6d) The magnetic field is assumed to be uniform and radially oriented at the Sun. It is also time-invariant. If the solar wind is assumed to be infinitely conducting, show that the magnetic field lines are aligned with the spiral streamlines of the solar wind. What is the electric field in this rotating frame?

6e) For a stationary observer in the solar equatorial plane at a distance *r* from the Sun, the solar wind flows radially with a velocity U_{SW}. Define the transformation velocity from the frame rotating with the Sun. Describe the magnetic field lines in this frame. What is the electric field in this stationary frame?

6f) Deduce the equation of the magnetic field lines in the solar equatorial plane, using a (plane) polar coordinate system (r,ϕ) as illustrated in figure below. Calculate the angle θ_E of the magnetic field with the radial direction at the distance of the Earth. Calculate also the angle ϕ_0 of the origin at the Sun of the magnetic field line which intercepts the Earth.

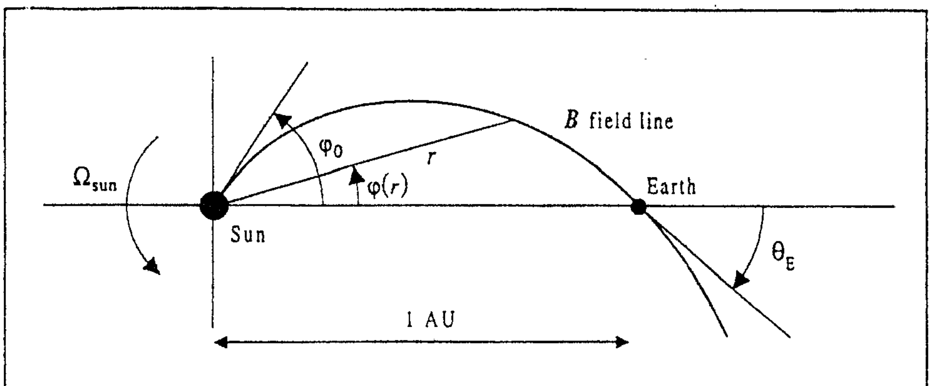

Fig. 1