Space Physics Problem Sheet 5 Solutions

Question 1:

1a) Let $\underline{j} = j_x \hat{\underline{x}} + j_y \hat{\underline{y}} + j_z \hat{\underline{z}}$, with $\underline{B} = B\hat{\underline{z}}$. Then

$$\underline{j} \times \underline{B} = j_y B \hat{\underline{x}} - j_x B \hat{\underline{y}} \quad \text{and} \quad (\underline{j}\times\underline{B})\times\underline{B} = -j_x B^2 \hat{\underline{x}} - j_y B^2 \hat{\underline{y}}$$

$$= -B^2 (j_x \hat{\underline{x}} + j_y \hat{\underline{y}}) = -B^2 \underline{j}_\perp$$

(Note, $\hat{\underline{x}} \times \hat{\underline{y}} = \hat{\underline{z}}$, $\hat{\underline{y}} \times \hat{\underline{z}} = \hat{\underline{x}}$, $\hat{\underline{z}} \times \hat{\underline{x}} = \hat{\underline{y}}$, and of course $\hat{\underline{x}} \times \hat{\underline{x}} = \hat{\underline{y}} \times \hat{\underline{y}} = \hat{\underline{z}} \times \hat{\underline{z}} = 0$)

1b) Taking the cross product of equation (1) with $\underline{B}$ gives:

$$\underline{j}\times\underline{B} = \sigma_0 (\underline{E}\times\underline{B}) - \frac{\sigma_0}{n_e e}(\underline{j}\times\underline{B})\times\underline{B} = \sigma_0(\underline{E}\times\underline{B}) + \frac{\sigma_0 B^2}{n_e e}\underline{j}_\perp$$

Substituting into equation (1) gives:

$$\underline{j} = \sigma_0 \underline{E} - \frac{\sigma_0}{n_e e}\left[\sigma_0 (\underline{E}\times\underline{B}) + \frac{\sigma_0 B^2}{n_e e}\underline{j}_\perp\right]$$

Taking the components perpendicular to $\underline{B}$ on the two sides of this equation gives:

$$\underline{j}_\perp = \sigma_0 \underline{E}_\perp - \frac{\sigma_0^2}{n_e e}(\underline{E}_\perp \times \underline{B}) - \frac{\sigma_0^2 B^2}{n_e^2 e^2}\underline{j}_\perp$$

Grouping the terms in $\underline{j}_\perp$ on the L.H.S of the equation, and dividing through by the coefficient of this term gives:

$$\underline{j}_\perp = \frac{\sigma_0}{1 + \frac{\sigma_0^2 B^2}{n_e^2 e^2}}\underline{E}_\perp - \frac{\sigma_0^2}{n_e e}\,\frac{1}{1 + \frac{\sigma_0^2 B^2}{n_e^2 e^2}}(\underline{E}_\perp \times \underline{B}) \quad (2)$$

1c) Using the definitions $\sigma_0 = \dfrac{n_e e^2}{m_e \nu_c}$ and $\Omega_{ce} = \dfrac{-eB}{m_e}$ we can write

$$\frac{\sigma_0^2 B^2}{n e^2 e^2} = \frac{n e^2 e^4}{m e^2 \nu_c^2}\frac{B^2}{n e^2 e^2} = \frac{1}{\nu_c^2}\frac{e^2 B^2}{m e^2} = \frac{\Omega_{ce}^2}{\nu_c^2}$$

and $$\frac{\nu_c^2 \sigma_0^2}{n e e} = \nu_c^2 \sigma_0 \frac{\sigma_0}{n e e} = \frac{\nu_c^2 \sigma_0 e}{m e \nu_c} = \nu_c \sigma_0 \frac{\Omega_{ce}}{B}$$

Using these, equation (2) can be put into the form required:

$$\underline{j}_\perp = \frac{\nu_c^2}{\nu_c^2 + \Omega_{ce}^2}\sigma_0 \underline{E}_\perp + \frac{\nu_c \Omega_{ce}}{\nu_c^2 + \Omega_{ce}^2}\sigma_0 \frac{\underline{E}_\perp \times \underline{B}}{B} \qquad (3)$$

1d) The second term on the RHS of equation (1) is a vector $\perp$ to $\underline{B}$, so only the parallel component of the first term contributes to the parallel component of the current density $\underline{j}_{\parallel} = \sigma_0 \underline{E}_{\parallel}$.

This together with equation (3), add up to the total current density:

$$\underline{j} = \underline{j}_{\parallel} + \underline{j}_\perp \quad \text{which yields equation (4),}$$

$$\underline{j} = \sigma_0 \underline{E}_{\parallel} + \sigma_P \underline{E}_\perp + \sigma_H \frac{\underline{E}_\perp \times \underline{B}}{B} \qquad (4)$$

where the Pedersen conductivity is $\sigma_P = \frac{\nu_c^2}{\nu_c^2 + \Omega_{ce}^2}\sigma_0$ and

the Hall conductivity is $\sigma_H = \frac{\nu_c \Omega_{ce}}{\nu_c^2 + \Omega_{ce}^2}\sigma_0$.

Hence the first term in (4) is a current parallel to $\underline{B}$, field aligned currents, the second term is a current $\perp$ to $\underline{B}$ and $\parallel$ to $\underline{E}_\perp$ known as the Pedersen current, and the third term is a current $\perp$ to both $\underline{E}_\perp$ and $\underline{B}$, known as the Hall current.

Question 2:

2a) The magnetic field pressure is given by:

$$p_B = \frac{B^2}{2\mu_0} = \frac{B^2}{2\mu_0}\tanh^2\left(\frac{z}{L}\right),$$ so that the

total pressure is: $$p_{tot} = p + p_B$$

$$= p_0 \operatorname{sech}^2\left(\frac{z}{L}\right) + \frac{B_0^2}{2\mu_0}\tanh^2\left(\frac{z}{L}\right)$$

2b) If $p_0 = \frac{B_0^2}{2\mu_0}$, then we can write:

$$p_{tot} = p_0 \operatorname{sech}^2\left(\frac{z}{L}\right) + p_0 \tanh^2\left(\frac{z}{L}\right)$$

$$= p_0\left[\operatorname{sech}^2\left(\frac{z}{L}\right) + \tanh^2\left(\frac{z}{L}\right)\right] = p_0 = \frac{B_0^2}{2\mu_0},$$

because of the identity $\operatorname{sech}^2(u) + \tanh^2(u) = 1$, (Verify!), therefore p_{tot} = constant.

The plasma β is defined as $$\beta = \frac{p}{B^2/2\mu_0}$$

$$= \frac{2\mu_0 p_0}{B_0^2}\frac{\operatorname{sech}^2(z/L)}{\tanh^2(z/L)} = \frac{1}{\sinh^2(z/L)}$$

For $z=0$, $\sinh^2\left(\frac{z}{L}\right) = 0$, so that $\beta \to \infty$. For $\left|\frac{z}{L}\right| \gg 1$, $\sinh^2\left(\frac{z}{L}\right) \gg 1$, so that $\beta \ll 1$. At and near the centre of the current sheet, the plasma pressure dominates; but as we move away from the centre, the magnetic field pressure increases and dominates over the plasma pressure.

2c) Use Ampere's law

$$\underline{j} = \frac{1}{\mu_0}\nabla \times \underline{B},$$ this gives:

$$\underline{j} = \frac{B_0}{\mu_0 L}\operatorname{sech}^2\left(\frac{z}{L}\right)\hat{y}$$

2d) Calculate first,

$$\underline{j} \times \underline{B} = -\frac{B_0^2}{\mu_0 L} \operatorname{sech}^2\left(\frac{z}{L}\right) \tanh\left(\frac{z}{L}\right) \hat{\underline{z}}$$

and then $\nabla p = \hat{\underline{z}} \frac{dp}{dz}$ which gives:

$$\nabla p = -\frac{2p_0}{L} \operatorname{sech}^2\left(\frac{z}{L}\right) \tanh\left(\frac{z}{L}\right) \hat{\underline{z}}$$

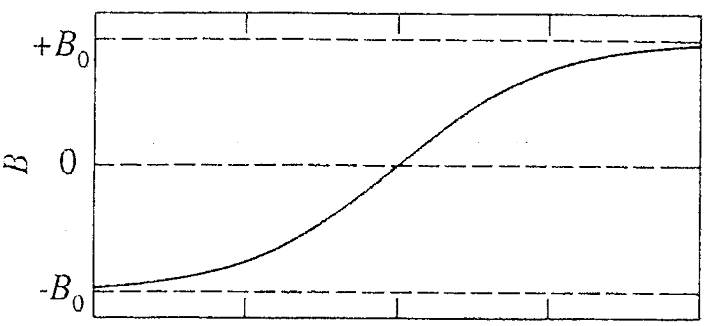
and as $p_0 = \frac{B_0^2}{2\mu_0}$, we get the force balance equation: $\underline{j} \times \underline{B} \overset{!}{=} \nabla p$

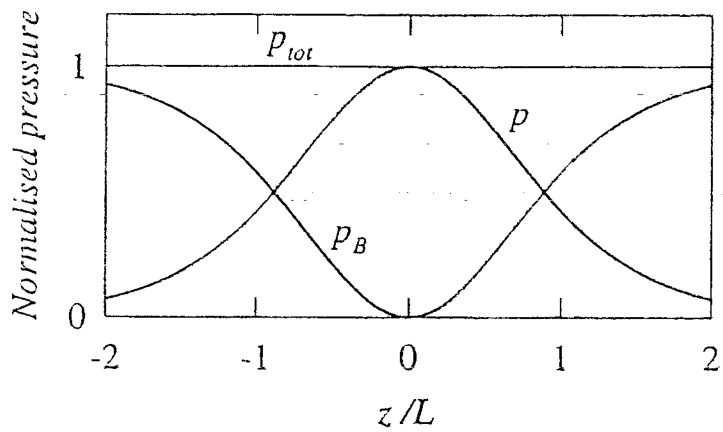
(Note, the functions used in this problem have the following definitions and properties:

$$\sinh\left(\frac{z}{L}\right) = \frac{e^{z/L} - e^{-z/L}}{2}, \quad \cosh(z/L) = \frac{e^{z/L} + e^{-z/L}}{2}$$

$$\tanh\left(\frac{z}{L}\right) = \frac{\sinh(z/L)}{\cosh(z/L)}, \quad \operatorname{sech}(z/L) = \frac{1}{\cosh(z/L)}$$

and the magnetic field and pressure behaviour as a function of (z/L) are shown below.





Question 3:

3a) From the values given, the flow speed in the tail lobe is

$$v_E = \frac{E}{B} = \frac{2\times10^{-4}}{2\times10^{-9}}\ ms^{-1} = 10^4\ ms^{-1} = 10\ kms^{-1}$$

The time to flow from the upper or lower tail lobe magnetopause to the current sheet at the centre of the tail (a distance of $L \sim 20 R_E = 1.3\times10^5$ km) is then:

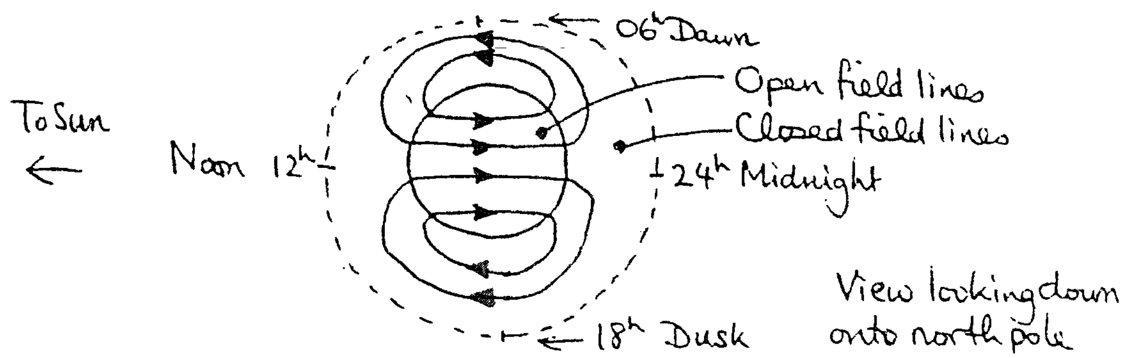
$$\frac{L}{v_E} = 1.3\times10^4\ s = 3.6\ \text{hours}.$$

3b) The length of the tail is hence given by the time a field line remains open times the solar wind speed, that is, the product

$$1.3\times10^4 \times 400\ km = 5.2\times10^6\ km = 812\ R_E.$$

The length of the tail is hence much greater than the distance to the magnetopause on the dayside of the Earth ($\sim 10 R_E$).

3c) The flow will be <u>away from the Sun</u> in the region of open field lines, and <u>towards the Sun</u> on closed field lines on the dawn - dusk meridian. Thus the flow streamlines form a twin-vortex as shown in the diagram below, as has been confirmed by spacecraft and radar observations.



Question 4:

4a) We use the formula: $\frac{R_{MP}}{R_P} = \left(\frac{2 B_{eq}^2}{\mu_0 n_{sw} m_p v_{sw}^2} \right)^{1/6}$

where $\rho_{sw} = n_{sw} m_p$.

The values of the various variables are as follows:

$B_{eq} = 31,000 \, nT = 31,000 \times 10^{-9} \, T = 31 \times 10^{-6} \, T \; (Wb\,m^{-2})$

$\mu_0 = 4\pi \times 10^{-7} \, H m^{-1} = 4\pi \times 10^{-7} \, \frac{Vs}{Am}$

$n_{sw} = 7 \, cm^{-3} = 7 \times 10^{6} \, m^{-3}$

$m_p = 1.6726 \times 10^{-27} \, kg$

$v_{sw} = 400 \, kms^{-1} = 400 \times 10^3 \, ms^{-1} = 4 \times 10^5 \, ms^{-1}$

where we have written all the variables in SI units.

Hence $\frac{R_{MP}}{R_P} = \left[\frac{2 \times (31 \times 10^{-6})^2}{4\pi \times 10^{-7} \times 7 \times 10^{6} \times 1.6726 \times 10^{-27} \times (4 \times 10^5)^2} \right]^{1/6}$

$= \left(\frac{1.922 \times 10^{-9}}{2.354 \times 10^{-15}} \right)^{1/6} = (816457.2)^{1/6}$

$= 9.7$

or $R_{MP} = 9.7 \, R_E$

4b) If v_{sw} is doubled, $v_{sw} = 800 \, kms^{-1}$, then a factor of $\left(\frac{1}{2^2}\right)^{1/6}$ is introduced, ie $\left(\frac{1}{2}\right)^{1/3}$, so that

$\frac{R_{MP}}{R_P} = 0.7937 \times 9.7 = 7.7$

Clearly the magnetopause moves closer to the Earth if the solar wind velocity, and at equal number density, its pressure is increased.

4c) The surface current density on the magnetopause boundary (is from Ampere's law)

$$J_{MP} = \frac{B_{MP}}{\mu_0},$$

where $B_{MP} = 2B_{eq}\left(\frac{R_P}{R_{MP}}\right)^3$, so that

$$J_{MP} = \frac{2B_{eq}}{\mu_0}\left(\frac{R_P}{R_{MP}}\right)^3, \text{ with } \frac{2B_{eq}}{\mu_0} = 49.338$$

in SI units.

In the first case for $v_{sw} = 400\ \mathrm{km\,s^{-1}}$,

$$J_{MP} = 49.338 \times \left(\frac{1}{9.7}\right)^3 = 0.054\ \mathrm{A\,m^{-1}}$$

In the second case for $v_{sw} = 800\ \mathrm{km\,s^{-1}}$,

$$J_{MP} = 49.338 \times \left(\frac{1}{7.7}\right)^3 = 0.108\ \mathrm{A\,m^{-1}}$$

Note, that in the second case, the current density is twice that in the first case. More surface current is needed in the magnetopause to compress the Earth's dipole field.